LOCAL SYSTEMS ON PROPER ALGEBRAIC V-MANIFOLDS

CARLOS T. SIMPSON

Abstract. We use coverings by smooth projective varieties then apply nonabelian Hodge techniques to study the topology of proper Deligne-Mumford stacks as well as more general simplicial varieties.

1. Introduction

This paper originated with the project of trying to understand how the techniques of harmonic theory and moduli spaces would apply to local systems over smooth proper Deligne-Mumford stacks.

The subject of DM-stacks has a rich history. The Kawamata-Viehweg vanishing theorem [57] [108] was originally proven by techniques involving cyclic or polycyclic Galois coverings of a smooth projective variety, ramified over a divisor with simple normal crossings. In current-day terms, Matsuki and Olsson have explained it as an instance of Kodaira vanishing over a root stack [68]. Viewed in this light, the vanishing theorem could be considered as one of the first major results about usual varieties where the geometry of Deligne-Mumford stacks plays a crucial role.

The coverings which appear in the original proofs may be viewed as varieties covering the DM-stack. We will take up this idea here to say, in Theorem 5.4, that any smooth proper DM-stack X is covered by a map $\phi : Z \to X$ from a disjoint union of smooth projective varieties such that every point downstairs $x \in X$ admits at least one point $z \in \phi^{-1}(x) \subset Z$ where ϕ is etale. A technical contribution to this statement comes from the Chow lemma of Gruson and Raynaud [85]. Proper coverings of stacks by schemes, with essentially similar constructions, have been considered by many authors, see for example [62], [83], [81].

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The goal here is to use these covering varieties Z to understand local systems on X. In order to do this, it is natural to look next at $Z \times_X Z$, but then resolve its singularities to get a smooth variety Z_1. This is the beginning of a simplicial resolution

$$Z_1 \rightrightarrows Z = Z_0 \to X$$

and standard constructions allow it to be completed to a full one. Such simplicial resolutions were used by Deligne for Hodge theory on singular varieties [33], and would seem to represent interesting topological objects in their own right. So we expand the level of generality by usually looking at a simplicial scheme $Z_\bullet$ such that the components Z_k are smooth projective varieties. In the differentiable category, these objects have been considered in [39] and [55]. No further topological generality would be gained by looking at simplicial objects whose levels are proper algebraic spaces.

The various moduli stacks of local systems on X may now be expressed as limits of the moduli stacks for the Z_k, Proposition 6.2. The moduli stacks admit universal categorical quotients which are the various versions of the character variety [65] of representations up to conjugacy. A natural question is to what extent these moduli stacks and their character varieties behave like in the smooth projective case.

After considering the general theory of moduli of local systems, we would like to use the covering varieties to do nonabelian harmonic analysis over the stack. In fact it turns out that we just have to apply the classical theory at each level of the simplicial variety. The surjectivity of the etale locus of the coverings allows us to interpret the result in terms of harmonic bundles on the original stack.

Our discussion of nonabelian harmonic theory on stacks adds to a subject which has already been treated by several authors [12] [36] [45] [103], and for the case of root stacks it is closely related to harmonic theory for parabolic bundles [15] [29] [64] [74] [70] [71] [84] [100]. The relationship between local systems and ramified covers can be related to the Chern class calculations of Esnault and Viehweg in [41], going back also to [80], and related formulae involving parabolic and orbifold bundles were studied in [53] [54]. This subject also connects with Viehweg's recent works such as [109] [110], since Shimura varieties are best considered as DM-stacks and indeed symmetric spaces were a main part of the original motivation for the notion of V-manifold [89] [90] which appears in our title. The other main motivation came from $\mathcal{M}_g$ [34], but as Campana has pointed out [26], local systems on orbifolds play an important part in the theory of moduli of more general varieties too. Examples over stacks locally of ADE-discriminant type up to

dimension 6 have been constructed in [72]. See [105] for a classification of orbifold structures over $\mathbb{P}^2$. Fascinating new examples have arisen with the notion of "twisted curves", see [27] for references.

A general simplicial scheme can have a pretty arbitrary topological type, for example any simplicial set with X_k finite for each k qualifies. For a general $X_\bullet$ one should therefore modify the kind of question being asked—not which topological types can occur, but rather how the topologies of the X_k interact with the full topological type of $|X_\bullet|$. This is a very interesting question closely related to the notion of nonabelian weight filtration.

The role of the weight filtration is illustrated by looking at the mixed Hodge structure on the complete local ring of the space of representations of π_1, Proposition 7.13, generalizing the recent paper [42]. A somewhat delicate point to beware of is the choice of basepoints. Even though we only need to use the representation spaces for the first two pieces of a simplicial resolution Z_0 and Z_1, the example 2.4 of three planes meeting in a point readily shows that one should be sure to choose basepoints meeting all components of Z_2.

Whereas the intervention of the weight filtration is to be expected in a general singular situation, one hopes that some kind of purity would hold for smooth proper DM-stacks.

For this, we can notice that the simplicial resolutions $Z_\bullet \to X$ arising for smooth Deligne-Mumford stacks have the nice property that the image of $\pi_1(Z_0)$ is of finite index in $\pi_1(X)$ (Condition 8.1). This guarantees that $\pi_1(X)$ doesn't include loops which jump from one place to another in Z_0 by going through the space Z_1 of 1-simplices. This condition allows us to recover much of the theory of moduli of local systems.

The finite-index condition holds for simplicial hyperresolutions of singular varieties, whenever the singularities are normal or indeed geometrically unibranched. The phenomenon we are trying to avoid is loops going through singular points and jumping from one branch to the other. We can therefore make the essentially straightforward observation that much of the theory known for the smooth projective case applies also to geometrically unibranched varieties, and in fact—combining the two examples—to geometrically unibranched DM-stacks.

Some of the main properties are Hitchin's hyperkähler structure, Theorem 8.8 and the continuous action of $\mathbb{C}^*$ whose fixed points are variations of Hodge structure, Lemma 7.6 and Corollary 8.9. These results all lead to restrictions on which groups can occur as fundamental groups of proper geometrically unibranched DM-stacks.

We take the opportunity to explain how Deligne's theory of [33] applies to get mixed twistor structures on the cohomology of semisimple local systems. Poincaré duality implies that these mixed structures are pure in the case of a smooth proper DM-stack.

Near the end of the paper, we discuss some constructions involving finite group actions, constructions which motivate the passage from smooth projective varieties to DM-stacks. If Φ is a finite group acting on a smooth projective X then the stack quotient of the moduli stack $\mathcal{M}(X,G)/\!/\Phi$ may be interpreted as a piece of the moduli stack of H-local systems on the DM-stack quotient $Y = X/\!/\Phi$ for a suitable group H (Corollary 11.5). In the last section of the paper, we answer a question posed by D. Toledo many years ago, showing that any group can be the fundamental group of an irreducible variety.

Conventions: All schemes are separated and of finite type over the field $\mathbb{C}$ of complex numbers.

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2. The topology of simplicial schemes

Let Δ be the category of nonempty finite linearly ordered sets denoted $[n] = \{0, \ldots, n\}$. A simplicial object in a category $\mathcal{C}$ is a functor $Y_\bullet : \Delta^o \to \mathcal{C}$, with levels denoted $Y_k := Y_\bullet([k])$. Following [33] an augmented simplicial object is a simplicial object $Y_\bullet$ together with another object $S \in \mathcal{C}$ and a natural transformation $p : Y_\bullet \to S$ from $Y_\bullet$ to the constant simplicial object with values S. This may also be considered as a functor $(\Delta \cup \{[-1]\})^o \to \mathcal{C}$ where $[-1] = \emptyset$ is the empty linearly ordered set, with $Y_{-1} = S$. We usually write such an object as $Y_\bullet \to S$, thinking of $\mathcal{C}$ as being contained in the category of simplicial $\mathcal{C}$-objects by the constant-object functor.

If $\mathcal{C} = Top$ or $\mathcal{C} = \mathcal{K}$ where $\mathcal{K} = Hom(\Delta^o, Sets)$ is the Kan-Quillen model category of simplicial sets, then a simplicial $\mathcal{C}$-object will be called a simplicial space. A simplicial space $Y_\bullet$ has a topological realization denoted $|Y_\bullet|$ which is a space, defined as the quotient space

of

$$\coprod_{k \in \Delta} Y_k \times R^k$$

by the relation $(\phi^*(y), r) \sim (y, \phi_* r)$ whenever $y \in Y_k, r \in R^m$ and $\phi : [m] \to [k]$ is a morphism in Δ. Here R^k are the standard k-simplices, fitting together into a cosimplicial space. The fat realization $\|X_\bullet\|$ is defined in the same way but using only the injective maps in Δ. For $\mathcal{C} = Top$ some cofibrancy conditions [77] must be imposed in order to have a homotopy equivalence $\|X_\bullet\| \xrightarrow{\sim} |X|$; these conditions are automatic for $\mathcal{C} = \mathcal{K}$, and also hold when $X_\bullet$ is s-split, so they will be tacitly assumed in all statements.

Suppose now $X_\bullet$ is a simplicial scheme. Then applying the usual functor to underlying topological spaces levelwise we obtain a simplicial space $X_\bullet^{\mathrm{top}}$ whose levels are the $(X_k)^{\mathrm{top}}$. The topological realization is a topological space

$$|X_\bullet| := |X_\bullet^{\mathrm{top}}|.$$

These spaces will be the main objects of our study.

A simplicial scheme or space has split degeneracies, or is s-split in Deligne's terminology [33], if each X_m is a disjoint union given by the degeneracy maps

$$X_m = X_m^N \sqcup \coprod_{k<m, m \twoheadrightarrow k} X_k^N.$$

The first term is $X_0^N = X_0$. We usually assume this condition, which also implies the cofibrancy conditions refered to above.

A local system on a simplicial space $Y_\bullet$ consists of a collection $L_\bullet = \{L_k\}$ of local systems L_k on Y_k, together with isomorphisms $\phi^*(L_k) \cong L_m$ whenever $\phi : [k] \to [m]$ induces $Y_m \to Y_k$, and these isomorphisms should satisfy the natural compatibility conditions as well as being the identity when ϕ is. This applies to local systems of abelian groups, vector spaces or modules over a ring, but also to local systems of sets and hence to G-torsors which are local systems of G-principal homogeneous sets.

We generally assume that our spaces are good enough that local systems correspond to representations of π_1. This assumption holds for the underlying topological spaces of schemes for example, but also for the realizations of simplicial spaces which levelwise are good enough and satisfy the required cofibrancy conditions.

Reflecting the fact that realizations and fat realizations are homotopy equivalent, the notion of local system is equivalent to the analogous notion defined using only the injective maps $\phi : [k] \hookrightarrow [m]$.

If $L_\bullet$ is a local system of abelian groups on $Y_\bullet$, we can choose a compatible system of injective resolutions $\mathcal{F}^\bullet_k$ of L_k over Y_k. Taking sections gives a simplicial complex of abelian groups whose total complex $t(\mathcal{F}^\bullet_\bullet(Y_\bullet))$ is defined by

$$t(\mathcal{F}^\bullet_\bullet(Y_\bullet))^i = \bigoplus_{k+j=i} \mathcal{F}^j_k(Y_k),$$

with differential using the alternating sign of face maps. The cohomology $H^i(Y_\bullet, L_\bullet)$ is defined to be the cohomology of this total complex. It is independent of the choice of resolution.

Lemma 2.1. *Suppose L is a local system over $|Y_\bullet|$. For each k, let L_k be the pushdown along $Y_k \times R^k \to Y_k$ of the restriction of L. Then the L_k fit together to form a local system $L_\bullet$ on $Y_\bullet$ and this construction establishes an equivalence of categories between local systems on $|Y_\bullet|$ and local systems on $Y_\bullet$. If L is a local system of abelian groups on $|Y_\bullet|$ and $L_\bullet$ the corresponding local system on $Y_\bullet$ then there is a natural isomorphism $H^i(|Y_\bullet|, L) \cong H^i(Y_\bullet, L_\bullet)$.*

Corollary 2.2. *The groupoid of G-torsors on $|Y_\bullet|$, denoted $H^1(|Y_\bullet|, G)$, is the 2-limit of the Δ-diagram of groupoids $k \mapsto H^1(Y_k, G)$.*

This generalizes to higher nonabelian cohomology: if T is an n-groupoid then $\mathrm{Hom}(\Pi_n(|Y_\bullet|), T)$ is the $n+1$-limit of the functor from Δ to $nGPD$ given by $k \mapsto \mathrm{Hom}(\Pi_n(Y_k), T)$.

For basepoints, rather than simply choosing a single one, it is often necessary to consider a map from a simplicial set. Suppose $U_\bullet$ is a simplicial set with a map $U_\bullet \to Y_\bullet$. For each k we obtain a collection of points U_k mapping to Y_k; it is more convenient not to require the map to be injective. The realization $|U_\bullet|$ is just the usual realization of the simplicial set, and we obtain a map $|U_\bullet| \to |Y_\bullet|$.

Say that $U_\bullet$ is 0-truncated if the realization $|U_\bullet|$ is a 0-truncated space, i.e. its homotopy groups vanish in degrees $i \geq 1$. Equivalently, it is a disjoint union of contractible pieces. A simplicial basepoint is a map $U_\bullet \to Y_\bullet$ such that $U_\bullet$ is a 0-truncated simplicial set with each U_k finite.

We mainly consider such a $U_\bullet$ which is a finite disjoint union of standard simplices. Let $h([k])$ denote the representable simplicial set represented by $[k] \in \Delta$, thus $h([k])_m = \Delta([m], [k])$. It is contractible. Suppose given a point $y \in Y_k$; this induces a map $h([k]) \to Y_\bullet$, and furthermore any map is induced from a point $y \in Y_k$ in that way. As notation, write $\langle y \rangle := h([k])$ together with the given map to $Y_\bullet$. If $\{y_i\}$ is a collection of nondegenerate points $y_i \in Y_{k_i}$ such that the $\langle y_i \rangle$ are

disjoint, their union

$$U_\bullet := \coprod_i \langle y_i \rangle \to Y_\bullet$$

is a simplicial basepoint. It is often convenient to look at $v_0 y_i \in Y_0$, the 0-th vertex of y_i, corresponding to $[0] \subset [k_i]$. It realizes to a point also denoted $v_0 y_i \in |\langle y_i \rangle| \subset |Y_\bullet|$.

Suppose $L_\bullet$ is a local system on $Y_\bullet$ corresponding to L on $|Y_\bullet|$. If $U_\bullet \to Y_\bullet$ is a map from a simplicial set, then the restriction of $L_\bullet$ to $U_\bullet$ is a local system on the realization $|U_\bullet|$. In particular, if $U_\bullet$ is 0-truncated, then the restriction is trivializable on each contractible connected component of $|U_\bullet|$, and a choice of trivialization is equivalent to a choice of trivialization over any point of this component.

Apply this to a simplicial basepoint $U_\bullet = \coprod_i \{\langle y_i \rangle\}$, with $y_i \in Y_{k_i}$. The inclusion of the 0-th vertex into the standard simplex R^{k_i} yields an isomorphism

$$L_{k_i}(y_i) \cong L(v_0 y_i).$$

A trivialization of $L_\bullet$ restricted to $U_\bullet$ is therefore the same thing as a collection of trivializations of $L_{k_i}(y_i)$ or a collection of trivializations of $L(v_0 y_i)$.

If $U \to Y$ is a map of spaces and G is a group, denote by $H^1(Y, U; G)$ the groupoid of G-torsors on Y together with trivializations of the pullbacks to U. If the image of U meets each connected component of Y then this groupoid is a discrete set.

If $U_\bullet \to Y_\bullet$ is simplicial basepoint, then we obtain a diagram

$$k \mapsto H^1(Y_k, U_k; G)$$

of groupoids.

Proposition 2.3. *Suppose that $U_\bullet \to Y_\bullet$ is a simplicial basepoint. Suppose G is a group. Suppose that U_k meets all the connected components of Y_k for $k = 0, 1, 2$. Then $H^1(|Y_\bullet|, |U_\bullet|; G)$ is the equalizer of the pair of face maps*

$$H^1(Y_0, U_0; G) \rightrightarrows H^1(Y_1, U_1; G).$$

Let $P := \pi_0(|U_\bullet|)$. Then G^P acts on this equalizer and the quotient groupoid is $H^1(Y_\bullet, G)$.

Proof. The cohomology 1-groupoid $H^1(|Y_\bullet|, |U_\bullet|; G)$ is the 2-limit of the family of cohomology groupoids $H^1(Y_k, U_k; G)$ indexed by $k \in \Delta$. This only depends on the initial part for $k = 0, 1, 2$, as will be explained later in Lemma 6.1. If U_k meets all components of Y_k for $k = 0, 1, 2$ then the groupoids are discrete, and the 2-limit is a 1-limit of a diagram of sets, which in turn is equal to the stated equalizer. $\square$

Notice that (Y_2, U_2) doesn't enter into the expression for the cohomology groupoid $H^1(|Y_\bullet|, |U_\bullet|; G)$. However, if U_2 doesn't meet all the connected components of Y_2 then the expression may not be true as the following example shows.

Example 2.4. *Let X be a singular variety, union of three coordinate planes meeting at the origin in $\mathbb{P}^3$. Let $Y_\bullet$ be the standard simplicial resolution [33] with Y_0 being the disjoint union of three planes, the nondegenerate part of Y_1 being the disjoint union of three lines, and the nondegenerate part of Y_2 being the origin.*

If U contains a basepoint on each of the double intersections but not at the origin, then the equalizer in the expression of Proposition 2.3 is different from $H^1(X, U, G)$.

To see this, let X' be the pyramid consisting of three copies of $\mathbb{P}^1 \times \mathbb{P}^1$ meeting along three disjoint lines. For a set of basepoints $U \subset X$ not containing the origin, one can choose a similar collection $U' \subset X'$ for which the expression of the equalizer in 2.3 is the same. In the case of X' there are no triple intersections so the equalizer expression is the correct one and it gives $H^1(X', U', G)$. However, $\pi_1(|X'|) = \mathbb{Z}$ whereas X was simply connected, so $H^1(X', U', G) \neq H^1(X, U, G)$.

3. Deligne-Mumford stacks

Let Sch denote the category of separated schemes of finite type over $\mathbb{C}$. Provided with the etale topology it becomes a site.

Classically, a 1-stack over Sch is viewed as a category fibered in groupoids $\mathcal{X} \to \mathrm{Sch}$, satisfying a descent condition. Recall that a fibered category can be strictified to a presheaf of 1-groupoids by setting $X(S)$ equal to the groupoid of sections $\mathrm{Sch}/S \to \mathcal{X}$. There is also a more topological approach.

Let $\mathbf{SP}$ denote the category of simplicial presheaves, with $\mathcal{W}$ defined as the class of Illusie weak equivalences. Let $\mathbf{SP}_1 \subset \mathbf{SP}$ be the subcategory of 1-truncated simplicial presheaves X, that is ones where $X(S)$ has $\pi_i = 0$ for $i \geq 2$.

Given a presheaf of 1-groupoids, the corresponding presheaf of nerves is in $\mathbf{SP}_1$. Conversely given a 1-truncated simplicial presheaf, we can look at the presheaf of Poincaré 1-groupoids. For speaking of 1-prestacks, these constructions, together with the strictification construction described above, set up an essential equivalence between the classical fibered-category point of view, and the category $\mathbf{SP}_1$.

Illusie weak equivalence defines a class of morphisms still denoted by $\mathcal{W}$ in $\mathbf{SP}_1$. The $\mathcal{W}$-local objects in $\mathbf{SP}_1$ correspond to presheaves of 1-groupoids or fibered categories which satisfy the descent condition to

be 1-stacks, see [49] for example. Denote by $\mathbf{SP}_{1,\mathrm{loc}}$ the subcategory of $\mathcal{W}$-local objects; one may equivalently take the subcategory of fibrant objects for either the projective or injective model structures.

Dwyer-Kan localization provides a simplicial or $(\infty, 1)$-category

$$\mathbf{St} := L_{DK}(\mathbf{SP}_{1,\mathrm{loc}}, \mathcal{W}) \sim L_{DK}(\mathbf{SP}_1, \mathcal{W})$$

of 1-stacks on Sch. It is 2-truncated, that is to say the mapping spaces are 1-truncated, so in Lurie's terminology it corresponds to a $(2, 1)$-category. This is a 2-category in which all 2-morphisms are invertible. This is the same as the classical 2-category of 1-stacks over the site Sch, a compatibility well-known particularly from Hollander's work [49].

The above viewpoint involving localization is useful for defining the topological realization of a stack. The topological realization functor on simplicial presheaves, considered in [98], [101], [38], is denoted

$$|\ | : \mathbf{SP} \to Top$$

where we are using Top as shorthand for the Kan-Quillen model category of simplicial sets. It sends Illusie weak equivalences to weak equivalences, so it passes to the Dwyer-Kan localizations. Let **TOP** denote the $(\infty, 1)$-category which is the Dwyer-Kan localization of Top by the weak equivalences. Then we get an $(\infty, 1)$-functor

$$|\ | : L_{DK}(\mathbf{SP}_{1,\mathrm{loc}}, \mathcal{W}) \to \mathbf{TOP}$$

which is written as a realization functor for stacks

$$|\ | : \mathbf{St} \to \mathbf{TOP}.$$

Note that $|X|$ is equivalent to the realization of any simplicial presheaf which is Illusie weak-equivalent to X. From this, follows the compatibility of realization with etale hypercoverings. If $Y_\bullet$ is a simplicial scheme, then since objects of Sch determine representable presheaves, we obtain a simplicial presheaf. An etale hypercovering of a stack X is a morphism in $L_{DK}(\mathbf{SP}, \mathcal{W})$

$$Y_\bullet \to X$$

such that the matching maps

$$Y_k \to \mathrm{match}_k(Y_\bullet \to X)$$

are coverings in the etale topology. Here the simplicial coordinate is included in $Y_\bullet$ but not in X to emphasize that we are considering this as an augmented simplicial object in $L_{DK}(\mathbf{SP}, \mathcal{W})$, but it may also be viewed as just a morphism in $L_{DK}(\mathbf{SP}, \mathcal{W})$. The fact that X is a stack rather than a scheme doesn't affect the definition of hypercovering, see Remark 5.1 below.

An etale hypercovering is, when viewed as a morphism of simplicial presheaves, an Illusie weak equivalence.

In this situation, $k \mapsto |Y_k|$ is a simplicial space denoted $|Y|_\bullet$. We have a weak equivalence of spaces

$$|(|Y|_\bullet)| \sim |X|.$$

In other words, the topological realization of X may be calculated by first choosing an etale hypercovering, then taking the associated simplicial space, and taking the topological realization of that in the sense used at the start of the paper. This brings us back to Noohi's construction of the topological realization of a stack [79], and similar constructions considered by Gepner, Henriques [44] and Ebert [40].

If $Z_\bullet$ is a simplicial object in **DMSt** then $k \mapsto |Z_k|$ is a simplicial space, whose realization also denoted $|Z_\bullet|$ is functorial in $Z_\bullet$. For a simplicial scheme this coincides up to weak equivalence with the realization defined previously. In particular, if $Z_\bullet \stackrel{a}{\to} X$ is a morphism from a simplicial scheme to a stack, considering the target as a constant simplicial object which has the same realization, we obtain a map of spaces

$$|Z_\bullet| \xrightarrow{|a|} |X|. \tag{3.1}$$

In the case of the etale hypercovering $Y_\bullet$ this is the weak equivalence considered above; we shall be interested in it for a proper surjective hypercovering.

A 1-stack X is a Deligne-Mumford (DM) stack if it has a presentation of the form $X = Z/R$ where Z is a separated scheme of finite type over $\mathbb{C}$, and $R \to Z \times Z$ is a groupoid in the category of schemes such that each projection $R \to Z$ is etale. For smooth DM-stacks, this notion is the algebraic analogue of Satake's notion of V-manifold [89] [90] or "orbifold", with the added feature that the generic stabilizer group can be nontrivial. But even if we start with a V-manifold, natural substacks can have nontrivial generic stabilizer so that possibility remains geometrically motivated and should be included.

The collection of DM-stacks naturally forms a 2-category which we denote by **DMSt**, a full sub-2-category of **St**. The 2-category structure comes about because one can have nontrivial natural automorphisms of morphisms $f : X \to Y$. This phenomenon occurs particularly if the automorphism group in Y at the general point of the image of f is nontrivial. Note however that if Y is a scheme or algebraic space, then maps from any stack to Y have no nontrivial automorphisms.

It is instructive to consider the case where $Y = V /\!/ G$ is a quotient stack of a scheme V by the action of a finite group G. In this case, a map

$X \to Y$ is a pair (T, ϕ) where $T \to X$ is a G-torsor and $\phi : T \to V$ is an equivariant map. An isomorphism between two maps $(T, \phi) \cong (T', \phi')$ is an isomorphism of G-torsors $u : T \cong T'$ such that $\phi' u = \phi$.

Following the previous discussion, let $\mathbf{SP}_{DM} \subset \mathbf{SP}_{1,\mathrm{loc}}$ denote the full subcategory of simplicial presheaves corresponding to 1-stacks which are Deligne-Mumford. Then

$$\mathbf{DMSt} := L_{DK}(\mathbf{SP}_{DM}, \mathcal{W})$$

is the $(\infty, 1)$-category defined by Dwyer-Kan localization along the Illusie weak equivalences (which, for $\mathcal{W}$-local objects, are the same thing as the objectwise weak equivalences of simplicial presheaves or, in a terminology more adapted to 1-stacks, objectwise equivalences of 1-groupoids). Again this is 2-truncated, i.e. it is really a $(2, 1)$-category, and we denote also by **DMSt** the same considered as a classical 2-category in which the 2-morphisms are invertible.

This 2-category has a 1-truncation $\tau_{\leq 1}\mathbf{DMSt}$. It is the category whose objects are DM-stacks and whose morphisms are equivalence classes of morphisms. The projection functor

$$\mathbf{DMSt} \to \tau_{\leq 1}\mathbf{DMSt}$$

does not have a section, as one can already see on examples of the form BG for a finite group G. Hence, when we speak of a "map between DM-stacks" it means a morphism of simplicial presheaves or fibered categories. Thus, by the "category of DM-stacks" we really mean either $\mathbf{SP}_{DM}$ or the more classical category whose objects are categories fibered in 1-groupoids over Sch. In these categories there will usually be several different morphisms representing the same equivalence class.

The 2-functor $\mathbf{DMSt} \to \mathbf{TOP}$ gives us some additional structure. Suppose X, Y are DM-stacks. Then $Hom_{\mathbf{DMSt}}(X, Y)$ is a groupoid, and its realization maps to the space $Hom_{\mathbf{TOP}}(|X|, |Y|)$. Given a map $X \to Y$, this gives a map of spaces from the classifying space of the finite group of natural automorphisms of f, to the the mapping space:

$$B(Aut_{\mathbf{DMSt}(X,Y)}(f)) \to Hom_{\mathbf{TOP}}(|X|, |Y|).$$

The first part of this structure is just the map of groups

$$Aut_{\mathbf{DMSt}(X,Y)}(f) \to \pi_1(Hom_{\mathbf{TOP}}(|X|, |Y|), |f|)$$

but the map of spaces contains extra structure which would be interesting to study further.

4. The structure of DM-stacks

One of the original goals of this work was to get information about the topology of DM-stacks. In preparation for the construction of

smooth projective covering varieties, we first recall some standard structural results. Many references are available: we have found [46] to be useful and concise, [103] discusses a wide range of topics, numerous papers of Olsson and other co-authors [81] ... provide invaluable viewpoints, and [2] is a guide to the extensive literature; in the future [31] will provide a definitive reference.

A closed substack is a morphism $Y \to X$ such that on any etale chart $Z_i \to X$, the fiber product $Y \times_X Z_i$ is a closed subscheme of Z_i. This amounts to specifying a closed substack on each chart, compatible with the glueing equivalence relation. The intersection of any number of closed substacks is again a closed substack. Notice, however, that a morphism from a point is not generally a closed substack, for example the only closed substacks of BG are $\emptyset$ and BG itself.

A Cartier divisor D on X is the specification for each etale chart $p : Z_p \to X$ of a Cartier divisor D_p on Z_p, such that if $Z_q \xrightarrow{f} Z_p \to X$ is a diagram of etale charts then $f^*(D_p) = D_q$. In this paper the word divisor will mean a Cartier divisor. For a scheme or an algebraic space this definition coincides with the usual one. If $f : X \to Y$ is a morphism of DM-stacks and D is a divisor on Y then, if no irreducible component of X maps into D we can define the pullback $f^*(D)$. The divisors D_p in the definition above are also the pullbacks $D_p = p^*(D)$. We say that D has normal crossings if for any etale chart $p : Z_p \to X$ the divisor $p*(D) = D_p$ has normal crossings. A divisor may be identified with a closed substack. The etale charts for the substack D are the D_{p_i} for etale charts p_i covering X.

A DM-stack X is separated if the diagonal map $X \to X \times X$ is proper, which is equivalent to a valuative criterion or also to saying that the map $R \to Z \times Z$ in the groupoid defining X is proper.

A DM-stack X is proper if and only if it is separated and satisfies the valuative criterion, saying that for any discrete valuation ring A with fraction field K and any map $Spec(K) \to X$ there exists an extension to a map $Spec(A') \to X$ where A' is the normalization of A in a finite extension K' of K. This is equivalent to the existence of a surjective covering map from a proper scheme [63] [81] [46].

Recall the results of Keel and Mori [58]. For any separated finite-type DM-stack X there exists an algebraic space $X^{\mathbf{c}}$ called the coarse moduli space together with a finite map $X \to X^{\mathbf{c}}$. It is universal for maps from X to a separated algebraic space, and furthermore if Y is an algebraic space mapping to $X^{\mathbf{c}}$ then $X \times_{X^{\mathbf{c}}} Y \to Y$ is also universal for maps to an algebraic space.

Locally over $X^{\mathbf{c}}$ in the etale topology, X is a quotient stack. Toen [103, Proposition 1.17] refers to Vistoli [112, Proof of 2.8] for this statement; see also Kresch [61].

The functorial resolution of singularities of Bierstone-Milman [9] and Villamayor [111] implies resolution of singularities for Deligne-Mumford stacks:

Proposition 4.1. *Suppose X is a reduced separated DM-stack of finite type over Spec$(\mathbb{C})$. Then there exists a surjective proper birational morphism $Z \to X$ of DM-stacks, an isomorphism over the dense Zariski open substack of smooth points of X, such that Z is smooth.*

Proof. Functoriality of the resolution procedure for etale morphisms means that the glueing procedure described in [10, §7.1], see also [111], extends to the case of etale open coverings of a DM-stack. □

One of the main constructions of Deligne-Mumford stacks is to look at the Cadman-Vistoli root stacks. Let X be a smooth projective variety, and $D = D_1 + \ldots + D_k$ a divisor with normal crossings broken up into its components D_i which are assumed to be irreducible and smooth. Fix a sequence of strictly positive integers $n_1, \ldots, n_k$. Cadman [25] defines and studies a stack $Z := X[\frac{D_1}{n_1}, \ldots, \frac{D_k}{n_k}]$ with a morphism $Z \to X$. Often we choose the same n for each component. Vistoli had also considered these stacks, see [1].

In a philosophical sense, the technique of root stacks may be traced back to Viehweg's use of cyclic coverings branched along a normal crossings divisor [108] and Kawamata's covering lemma [56]. This covering technique has been used by many authors since then; for a recent example see Urzúa [106].

The stack Z can be explicitly presented as a quotient stack locally in the Zariski topology of X, indeed the construction of etale charts in Cadman [25] actually gives a local quotient structure. Over a neighborhood in X where D_i have equations $f_i = 0$, the chart is the subvariety of $X \times \mathbb{A}^k$ given by $f_i = u_i^{n_i}$.

Within the local charts, one can remark that there is a standard divisor denoted $R = R_1 + \ldots + R_k$ in $X[\frac{D_1}{n_1}, \ldots, \frac{D_k}{n_k}]$, and $n_i \cdot R_i = p^*(D_i)$ where p is the projection from the Cadman stack back to X. In particular if all the n_i are the same n then $n \cdot = D$. Note also that R has normal crossings, as can be seen in the local charts.

Lemma 4.2. *Suppose $f : Y \to X$ is a finite Galois covering from a normal variety, unramified outside D, with Galois group Φ corresponding to a representation $\varphi : \pi_1(X - D) \to \Phi$. Then f lifts to a map $\tilde{f} : Y \to Z$ if and only if, for each point $x \in D_{i_1} \cap \cdots \cap D_{i_r}$ the kernel*

of the map from the local fundamental group

$$\mathbb{Z}^r \to \Phi$$

is contained in $n_{i_1}\mathbb{Z} \oplus \cdots \oplus n_{i_r}\mathbb{Z} \subset \mathbb{Z}^r$. *The map* $\tilde{f}$ *is an etale covering space if and only if equality holds for the kernel at each point* x. *In this case* Y *is smooth.*

The "Kawamata covering lemma" [56, Theorem 17] gives us projective varieties covering the root stack. I first learned about this kind of idea when reading Viehweg's paper [108], even though his technical approach, investigating further the singularities of purely cyclic coverings arising from the crossing points, is different from Kawamata's.

Lemma 4.3. *If* X *is a smooth variety with simple normal crossings divisor* $D = D_1 + \ldots + D_k$, *and if* $Z = X[\frac{D_1}{n_1}, \ldots, \frac{D_k}{n_k}]$ *is a root stack, then for any* $z \in Z$ *there exists a smooth variety* Y *with a finite, flat morphism* $r : Y \to Z$ *such that* r *is a finite etale covering over a neighborhood of* z.

Proof. Recall the procedure from [56]. For each divisor component D_i, choose a very ample divisor K_i such that $D_i + K_i$ is a multiple of n_i in $\mathrm{Pic}(X)$. Then choose representatives $K_i^j \sim K_i$, such that the full divisor

$$\mathcal{D}_K := \sum_i D_i + \sum_{i,j} K_i^j$$

has normal crossings. For each i, j there is a cyclic covering branched along D_i and K_i determined by choosing an n_i-th root of $D_i + K_i^j$. These coverings determine subgroups of $\pi_1(X - \mathcal{D}_K)$, and Kawamata shows (in a more algebraic notation) that if enough K_i^j are chosen for each i, then the intersection of all of these subgroups satisfies the conditions of Lemma 4.2. That gives a smooth variety Y branched over $\mathcal{D}_K$ and mapping to the root stack over $\mathcal{D}_K$. Composing with the projection

$$Y \to X[\ldots, \frac{D_i}{n_i}, \ldots, \frac{K_i^j}{n_i}, \ldots] \to X[\ldots, \frac{D_i}{n_i}, \ldots]$$

gives a finite flat map r. The K_i^j are chosen arbitrarily in very ample linear systems, so we can assume that they miss the given point z, in which case r will be etale over z. □

The Cadman-Vistoli root stack satisfies a good extension property for morphisms.

Proposition 4.4. *Suppose (X, D) is a smooth variety with normal crossings divisor as above. Suppose Y is an irreducible DM-stack with coarse moduli space $Y^{\mathbf{c}}$. Suppose given a diagram*

$$\begin{array}{ccc} X - D & \rightarrow & Y \\ \downarrow & & \downarrow \\ X & \rightarrow & Y^{\mathbf{c}}. \end{array}$$

Suppose n_i are strictly positive integers. A lifting over the root stack

$$\tilde{f} : X[\frac{D_1}{n_1}, \ldots, \frac{D_k}{n_k}] \rightarrow Y$$

fitting into commutative diagrams with the given maps, is unique up to unique isomorphism if it exists. Furthermore, there exists a choice of n_i such that a lifting exists.

Proof. Consider first the unicity statement when $n_i = 1$, i.e. for extensions to X. For this, we can localize in the etale topology over $Y^{\mathbf{c}}$. By Keel-Mori, this means that we can assume $Y = Z/\!/G$ for a finite group G acting on an algebraic space Z. The given map $X - D \rightarrow Z/\!/G$ corresponds to a pair (T, ϕ) where T is a G-torsor on $X - D$ and $\phi : T \rightarrow Z$ is G-equivariant. An extension to X consists of $(\overline{T}, \overline{\phi})$ where $\overline{T}$ is an extension of T to a G-torsor on X and $\overline{\phi}$ extends ϕ. Since X is smooth—indeed geometrically unibranched would be sufficient here, a preview of the phenomenon to be met in Theorem 8.4 later—the extension $\overline{T}$ is unique up to unique isomorphism, and of course $\overline{\phi}$ is unique since X contains no embedded points. This shows the unicity up to unique isomorphism for extensions from $X - D$ to X.

For extensions over a root stack, use local smooth charts for the root stack and unicity of the extension on these charts from the previous paragraph, to get unicity up to unique isomorphism for extensions

$$X[\frac{D_1}{n_1}, \ldots, \frac{D_k}{n_k}] \rightarrow Y.$$

Now to construct an extension, in view of the unicity, we can localize in the etale topology over X, hence we can also localize in the etale topology over $Y^{\mathbf{c}}$. Therefore assume $Y = Z/\!/G$ for a finite group G acting on an algebraic space Z. In this case $Y^{\mathbf{c}} = Z/G$ is the usual quotient.

The map $X - D \rightarrow Z/\!/G$ corresponds to a pair (T, ϕ) where T is a G-torsor on $X - D$ and $\phi : T \rightarrow Z$ is a G-equivariant map. The local fundamental group of $X - D$ near a point $x \in D_{i_1} \times \cdots \times D_{i_r}$ of D is of the form $\mathbb{Z}^r$, but G is finite so its action on T factors through a quotient of the form $\mathbb{Z}/n_{i_1} \times \mathbb{Z}/n_{i_r}$. Let n_i be a common multiple of

the integers appearing here for all points of D_i. Then T extends to a torsor over the root stack

$$\overline{T} \to X[\frac{D_1}{n_1}, \ldots, \frac{D_k}{n_k}].$$

Note that the total space of $\overline{T}$ itself is a smooth algebraic space, and the inverse image of the divisor is a divisor with normal crossings $R \subset \overline{T}$. It remains to extend ϕ. However, $\overline{T}$ is a normal space and $Z \to Z/G$ is a finite map. It follows that one can extend the given map $\phi : \overline{T} - R \to Z$ to a map $\overline{\phi} : \overline{T} \to Z$, from knowing that the extension $\overline{T} \to Z/G$ exists.

This may be seen on local affine charts: write $\overline{T} = \mathrm{Spec}(A)$, $Z = \mathrm{Spec}(B)$, so $\overline{T} - R = \mathrm{Spec}(A_g)$ where g is the function defining the divisor R, and $Z/G = \mathrm{Spec}(B^G)$. The extension $B^G \subset B$ is finite, and the map $B \to A_g$ sends B^G to A, it follows from normality of A that B maps into $A \subset A_g$, in a unique way hence G-equivariantly. This provides the required map $\overline{T} \to Z$ corresponding to an extension

$$X[\frac{D_1}{n_1}, \ldots, \frac{D_k}{n_k}] \to Z/\!/G.$$

Going back to Y and globalizing over $Y^{\mathbf{c}}$ gives the required extension to prove the lemma. $\square$

5. Proper surjective hypercoverings by smooth projective varieties

We use the notations of [33]. Suppose $X_\bullet \xrightarrow{a} S$ is an augmented simplicial scheme. For each $k \geq 0$, the coskeleton construction defines the matching object

$$\mathrm{match}_k(X_\bullet \to S) := \mathrm{csk}(\mathrm{sk}_{k-1} X_\bullet)_k$$

in Deligne's notation [33], and we have a natural "matching" map

$$X_k \to \mathrm{match}_k(X_\bullet \to S). \tag{5.1}$$

At $k = 0$ the matching map is just $X_0 \to S$ and at $k = 1$ it is $X_1 \to X_0 \times_S X_0$. For $k \geq 2$ the matching map is independent of the augmentation $X_0 \to S$.

A morphism $X_\bullet \to S$ is a proper surjective hypercovering if the matching maps are proper surjections, if $X_0 \to S$ is a proper surjection, and if $X_\bullet$ has split degeneracies. An etale hypercovering is given by requiring that the matching maps be coverings in the etale topology, i.e. admit sections etale-locally.

Remark 5.1. *The notion of proper surjective (resp. etale) hypercovering extends to the case where S is a separated DM-stack, indeed then*

$X_0 \times_S X_0$ is an algebraic space so the proper surjectivity of the matching maps at X_0 and X_1 are well-defined conditions.

It is a well-known fact that coverings of proper DM-stacks by projective varieties exist [62] [81]:

Lemma 5.2. *If X is a proper DM-stack then there exists a surjective proper map $Z \to X$ where Z is a smooth projective variety.*

Proof. One could first apply the general existence of proper coverings [81] and then apply the Chow lemma and resolve singularities; or alternatively, resolve first the singularities of X and then apply Theorem 5.4 below. □

Theorem 5.3. *A proper DM-stack X admits a proper surjective hypercovering with split degeneracies, by smooth projective varieties. Any two such hypercoverings can be topped off by a third one.*

Proof. Use the previous lemma to choose $Z_0 \to X$. Notice that $Z_0 \times_X Z_0 \to Z_0 \times Z_0$ is finite so $Z_0 \times_X Z_0$ is a projective variety. Continue from there using Deligne's technique [33]. □

Suppose $f : Z \to X$ is a morphism of DM stacks. We say that f is surjective where etale if, letting $Z' \subset Z$ be the open substack where f is etale, we have $Z' \to X$ surjective. This is equivalent to saying that any point $x \in X$ admits at least one lift $z \in Z$ such that f is etale near z. On the other hand f will not be etale or even finite in a neighborhood of a different point of the fiber over x.

The precise form of the Chow lemma proven by Raynaud and Gruson in [85] allows us to obtain a good form of the Chow lemma for smooth proper DM-stacks. This improves Deligne-Mumford's statement 4.12 of [34], or rather it gives a refined statement which, if they had given a proof, they would undoubtedly have proven along the way. See also Kresch-Vistoli [62], Olsson, and Starr [83], [81] for statements about existence of coverings.

The techniques of [85] have been applied in many similar situations. See Rydh [86] for a recent application, and de Jong [30] for a more classical utilisation. It is also interesting to note the extensive and detailed AMS review of [85] by Masaki Maruyama. Maehara refers to these techniques, while speaking of Kawamata-Viehweg coverings, in his paper [66].

Theorem 5.4. *Suppose X is a smooth and proper DM-stack of finite type. Then there exists a morphism $f : Z \to X$ such that f is surjective where etale and Z is a smooth projective variety.*

Proof. Fix a point $x \in X$. We will find $f : Z \to X$ with a lifting $z \in Z$ of x such that Z is smooth projective and f is etale at z.

Start with an etale neighborhood $p : U \to X$ with a lifting $u \in U$ of the point u, U an affine scheme of finite type over $\mathbb{C}$, and p an etale morphism. This exists by the definition of DM-stack. Note that U is smooth and quasiprojective.

Let $U \subset P$ be a completion to a smooth projective variety. Let $C \subset P \times X^{\mathbf{c}}$ be the closure of the graph of the map $U \to X^{\mathbf{c}}$ to the coarse moduli space. It is a proper algebraic space containing U as a Zariski open subset.

By resolution of singularities for algebraic spaces [111] [9] we can resolve the singularities of C without touching U, which gives a diagram of algebraic spaces

$$\begin{array}{ccc} U & \hookrightarrow & Y \\ & \searrow & \downarrow \\ & & C \end{array}$$

where Y is a smooth proper algebraic space and $D := Y - U$ is a divisor with normal crossings. Write $D = D_1 + \ldots + D_k$ and we may assume that the D_i are irreducible and smooth. Raynaud and Gruson [85, Cor. 5.7.14], refering also to Knutson [59], prove this version of the Chow lemma: if Y is a separated proper algebraic space and $U \subset Y$ is an open subset such that U is a quasiprojective variety, then there is a blow-up $\widetilde{Y} \to Y$ which is an isomorphism over U such that $\widetilde{Y}$ is projective. This means that after replacing Y by $\widetilde{Y}$ which is the same over U, then again resolving singularities of the complementary divisor, we can suppose that Y is projective.

We are now in the situation of Lemma 4.4 with a diagram

$$\begin{array}{ccc} U & \rightarrow & X \\ \downarrow & & \downarrow \\ Y & \rightarrow & X^{\mathbf{c}}. \end{array}$$

so there there is n, which for convenience can be assumed the same for all divisor components, such that the map extends over the root stack to a map

$$Y[\frac{D_1}{n}, \ldots, \frac{D_k}{n}] \to X.$$

We next note that there is a morphism from a projective scheme $Z \to Y[\frac{D_1}{n}, \ldots, \frac{D_k}{n}]$, which is finite and projects to a cover of Y ramified along a subset which misses u. This is exactly the covering lemma 4.3.

Hence, for any point $z \in Z$ lying over $u \in U$, the morphism $Z \to U$ is etale at z. Thus, we obtain a map $Z \to X$ as desired for the proof of the

theorem relative to one point. It suffices to take a finite disjoint union of such varieties Z in order to get a map surjective where etale. $\square$

There is a variant of this result which takes into account a divisor with normal crossings.

Proposition 5.5. *Suppose X is a smooth proper and separated DM-stack of finite type and $D \subset X$ a divisor with normal crossings. Then there exists a morphism $f : Z \to X$ such that f is surjective where etale, and Z is a smooth projective variety, such that $f^*(D)$ is a divisor whose associated reduced divisor has normal crossings.*

Proof. Left to the reader. $\square$

Question 5.6. *To what extent does the statement of Theorem 5.4 hold for DM-stacks which are not smooth?*

The above construction of covering spaces provides the starting point for the construction of a hypercovering. Follow the technique of [33] with a little extra care at the first stage to conserve a trace of the surjective-where-etale property.

Suppose X is a smooth proper DM stack. Let $p : Z \to X$ be a morphism from a smooth projective variety, such that X is covered by the open set Z' where p is etale.

Consider $R_1 := Z \times_X Z$ and $K_1 := Z \times_X Z \times_X Z$. These are algebraic spaces. Note that Z, R_1 and K_1 form the first part of a simplicial object. However, R_1 and K_1 are not smooth.

Lemma 5.7. *In fact R_1 and K_1 are projective.*

Proof. The map $R_1 \to Z \times Z$ is finite because X is a proper, whence separated DM stack. Since Z is projective, we get that R_1 is projective. The same holds for K_1. $\square$

Let $R' := (Z' \times_X Z) \cup (Z \times_X Z')$. Then $R' \subset R_1$ is a smooth open subset of R_1. It decomposes

$$R' = R'^{,N} \sqcup Z'$$

where $Z' \to R' \to Z \times_X Z$ is the diagonal. Notice that $Z' \to R'$ is a closed immersion, so it is one of the connected components of R'.

Let $R \to R_1$ be a resolution of singularities of the union of irreducible components of R_1 which meet R', and which is an isomorphism over $R' \subset R$. Choose in particular Z as the completion of the component $Z' \subset R'$. In this way, $R' \subset R$ is an open dense subset, and

$$R = R^N \sqcup Z.$$

This insures the split degeneracy condition for R.

The matching object at the next stage is the equalizer

$$M \to R \times R \times R \rightrightarrows Z \times Z \times Z$$

of the two maps sending (u, v, w) to $(\partial_0 u, \partial_0 v, \partial_1 w)$ and $(\partial_0 w, \partial_1 u, \partial_1 v)$ respectively.

Notice that M is projective. Rather than continue with a more careful choice such as was done with R (and which the reader is invited to do), we can just set $K^N \to M$ to be a resolution of singularities from a smooth projective variety, isomorphism over the smooth locus. Put

$$K_2 := K^N \sqcup R^N \sqcup R^N \sqcup Z.$$

Notice that $K' := Z' \times_X Z' \times_X Z'$ splits in the same way and we can choose a map $K' \to K_2$ respecting this decomposition. Let K be the union of components of K_2 containing K'. The map $K \to M$ is still surjective.

We now have a diagram

$$K \overset{\to}{\rightrightarrows} R \rightrightarrows Z \xrightarrow{f} X$$

plus the degeneracies going in the other direction, which looks like the beginning of an augmented simplicial object. In other words, the compositions which would be equal for a simplicial set, are also equal here. For maps into X one must replace "equality" by "isomorphism", and be careful about coherences. In the first three terms the elements are smooth projective varieties. Denote by ∂_0, ∂_1 the two maps from R to Z, and by ∂_{01}, ∂_{02} and ∂_{12} the three maps from K to R. Note that we have open dense subsets

$$Z' \times_X Z' \subset R' \subset R$$

and

$$Z' \times_X Z' \times_X Z' = K' \subset K,$$

and these open subsets form the beginning of the standard simplicial object for $Z' \to X$. The required equalities of maps $K \to Z$ follow because these open subsets are dense, and these serve to define three maps $v_0, v_1, v_2 : K \to Z$:

(5.2)

$$v_0 := \partial_0 \circ \partial_{01} = \partial_0 \circ \partial_{02}, \quad v_1 := \partial_1 \circ \partial_{01} = \partial_0 \circ \partial_{12}, \quad v_2 := \partial_1 \circ \partial_{02} = \partial_1 \circ \partial_{12}.$$

We have a natural isomorphism $\alpha : f \circ \partial_0 \cong f \circ \partial_1$, and the coherence conditions say that the hexagon made with three copies $\partial_{ij}^* \alpha$ and the three equalities above composed with f, commutes. Of course if X were an orbifold, that is a DM stack with trivial generic stabilizer, then generically surjective maps from an irreducible smooth variety into X wouldn't have any nontrivial isomorphisms, so in this case there would

have been no need to speak of α and the coherence condition. However, even in this case we will meet the hexagonal coherence condition when looking at bundles.

Theorem 5.8. *A smooth proper DM-stack X admits a proper surjective hypercovering with split degeneracies by smooth projective varieties, obtained by completing the partial simplicial object (Z, R, K) constructed above. Again, any two such hypercoverings can be topped off by a third one.*

Proof. For X smooth, starting from the first part constructed above, Deligne's technique [33] allows us to finish. □

Suppose $Z_\bullet$ is a simplicial scheme with smooth projective levels, and split degeneracies. Denote by (Z, R, K) the first three terms. Keep the notations from before, with two morphisms $\partial_0, \partial_1 : Z \to X$ and three morphisms $\partial_{ij} : R \to Z$ for $0 \le i < j \le 2$, and equalities (5.2) over K. The simplicial object then starts with $Z_0 = Z$, $Z_1 = R = R^N \sqcup Z$ and $Z_2 = K = K^N \sqcup R^N \sqcup R^N \sqcup Z$.

A descent datum for (Z, R, K) is a bundle E on Z, and an isomorphism $\varphi : \partial_0^* E \cong \partial_1^* E$ on R such that the hexagon formed by alternating the $\partial_{ij}^*\varphi$ with the equalities (5.2), commutes. A morphism between descent data $(E, \varphi) \to (E', \phi)$ is a morphism $E \to E'$ commuting with the isomorphisms. Given a bundle F on X, its pullback to Z is provided with a natural descent datum.

A descent datum for (Z, R, K) according to this definition is automatically compatible with the degeneracy map $s_0 : Z \to R$ in the sense that $s_0^*(\phi) = 1_E$. Indeed, if s_1 and s_2 denote the two degeneracy maps from R to K, then

$$\partial_1^* s_0^*(\phi) \circ \phi = s_2^* \partial_{12}^*(\phi) \circ s_2^* \partial_{01}^*(\phi) = s_2^* \partial_{02}^*(\phi) = \phi,$$

so $\partial_1^* s_0^*(\phi) = 1_{\partial_1^*(E)}$ since ϕ is invertible, so

$$s_0^*(\phi) = s_0^* \partial_1^* s_0^*(\phi) = s_0^*(1_{\partial_1^*(E)}) = 1_E.$$

Lemma 5.9. *The category of descent data for vector bundles (maybe with extra structure) over $Z_\bullet$ is equivalent to the category of explicit descent data on (Z, R, K).*

Proof. A descent datum on $Z_\bullet$ restricts in an obvious way to a descent datum on (Z, R, K). Suppose given a descent datum (E, φ) over (Z, R, K). The j-th vertex map $[0] \to [k]$ in Δ induces $v_j : Z_k \to Z_0 = Z$. A path of edges relating the i-th and j-th vertices in $[k]$ gives, using φ, an isomorphism of bundles between $v_i^*(E)$ and $v_j^*(E)$ on X_k. When two paths differ by the boundary of a 2-simplex, the equalities required

of φ imply that the two isomorphisms are the same. But any two paths can be connected by a sequence of transformations along boundaries of 2-simplices, so any two paths induce the same isomorphism. This canonically identifies all of the $v_j^*(E)$ to a unique bundle which can be called E_k. It is now easy to see that the E_k are naturally functorial for pullbacks along the simplicial maps $X_k \to X_m$. This constructs the essential inverse to the restriction functor. $\square$

Remark 5.10. *If (Z, R, K) is the start of a proper surjective hypercovering of a proper DM-stack X, and (E, φ) is a descent datum for a bundle or local system on (Z, R, K). Then φ determines a continuous isomorphism between $\mathrm{pr}_1^*(E)$ and $\mathrm{pr}_2^*(E)$ over $Z \times_X Z$.*

Proof. The map $R \to Z \times_X Z$ is proper and surjective, with φ defined over R. There is a map from $R \times_{(Z\times_X Z)} R$ to the matching object: given $r, r' \in R$ mapping to $(z, z') \in Z \times_X Z$, associate $(r, 1_{z'}, r')$ in the matching object. As K surjects to the matching object, and $\varphi(1_{z'}) = 1_{E(z')}$, the cocycle condition over K says that $\varphi(r') = \varphi(1_{z'}) \circ \varphi(r)$. Thus, φ descends as a continuous function to $Z \times_X Z$. $\square$

Return now to the hypothesis that X is a smooth DM-stack, and $Z_\bullet \to X$ is a proper surjective hypercovering with Z_k smooth projective, starting off with (Z, R, K) chosen according to the procedure described before Theorem 5.8. Thus $Z \to X$ is assumed to be surjective where etale, and R chosen as a completion of the smooth R' as above.

There is also a natural pullback functor from bundles on X to descent data on (Z, R, K).

When X is smooth, the extra information given by the surjective-where-etale property allows us to transfer analytic constructions from $Z_\bullet$ back to X, to get things like the definition and existence of harmonic metrics. It might be possible to descend these things along proper surjective hypercoverings too, and in that way get around Theorem 5.4 entirely, but that would require a much more detailed study of descent for bundles along proper surjective maps, a subject discussed in [13] [14].

Notice that $Z_\bullet$ may be chosen to contain an etale hypercovering as an open simplicial subvariety, but is not itself an etale hypercovering. This is because the places where the maps are not etale lead to singularities in the matching objects, so one needs to use resolution of singularities at each stage in order to have Z_k smooth. Nonetheless an explicit treatment of the first part of the resolution yields descent for bundles.

Lemma 5.11. *The category of bundles on X is naturally equivalent via this pullback functor to the category of descent data on $Z_\bullet$ or the partial simplicial object (Z, R, K).*

Proof. Recall that $Z' \subset Z$ is the Zariski dense open set over which the projection p is etale. By descent for the map $Z' \to X$, we obtain a bundle F over X, with an isomorphism $\psi' : p^*(F)|_{Z'} \cong E|_{Z'}$ compatible with the descent data over Z'.

Recall that $R' := Z' \times_X Z \cup Z \times_X Z'$ is a smooth open subset of R, which itself contains $Z' \times_X Z$ as an open subset. The descent datum yields an isomorphism $\partial_1^*(E|_{Z'}) \cong \partial_2^* E$ over $Z' \times_X Z$, but

$$\partial_1^*(E|_{Z'}) \cong \partial_1^*(p^*(F)|_{Z'}) \cong p_R^*(F)|_{Z' \times_X Z}$$

where $p_R : R \to X$ is the projection, whence

$$p_R^*(F)|_{Z' \times_X Z} \cong \partial_2^*(E)|_{Z' \times_X Z}.$$

The map $\partial_2 : Z' \times_X Z \to Z$ is an etale covering, and we are now given an isomorphism between $p^*(F)$ and E, locally with respect to this covering. Using the cocycle condition for the descent data over K, this isomorphism is the same as the previous one over Z'. The property of extending an isomorphism from a Zariski open set to the whole of Z is etale-local on the complementary closed subset, so this shows that our isomorphism extends to a global isomorphism $\psi : p^*(F) \cong E$ on Z. It is compatible with the given descent data since the open subsets Z' and R' are dense.

We have shown that the pullback functor from bundles to descent data is essentially surjective. To show that it is fully faithful, given a morphism between descent data it restricts to a morphism between descent data on $(Z', Z' \times_X Z')$ so descends to a morphism between bundles. □

Suppose $\lambda \in \mathbb{C}$. A λ-connection on a descent datum (E, φ) is a λ-connection ∇ on E, such that φ intertwines the pullbacks $\partial_0^*\nabla$ on $\partial_0^* E$ and $\partial_1^*\nabla$ on $\partial_1^* E$. This definition extends to objects defined over a base scheme S, for any $\lambda \in \Gamma(S, \mathcal{O}_S)$.

Lemma 5.12. *Suppose (E, φ) is the descent datum corresponding to a bundle F on X. Given a λ-connection ∇_E on (E, φ) there is a unique λ-connection ∇_F on F such that $f^*\nabla_F = \nabla_E$ via the isomorphism $f^*F \cong E$.*

Proof. As before, if Z were replaced by Z' this would be the classical etale descent. In particular, $\nabla_E|_{Z'}$ descends to a unique connection ∇_F on F. But now, $f^*\nabla_F$ and ∇_E are two λ-connections on E over the

smooth variety Z, with the same restriction to the dense open subset Z'. Therefore they are equal. Unicity of ∇_F follows by descent only over Z'. $\square$

We can similarly descend $\mathcal{C}^\infty$ vector bundles, hermitian metrics on them, and differential-geometric structures such as differential operators. It is left to the reader to formulate these statements.

Cohomological descent along proper surjective hyperresolutions was the main technique used by Deligne to apply Hodge theory to the topology of singular varieties. It is the main reason for looking at simplicial schemes, but the same techniques also apply to get proper surjective hyperresolutions for DM-stacks as stated in Theorem 5.8 above.

Proper surjective cohomological descent [88] [107] [33] then says that for any local system L on S,

$$H^i(S, L) \stackrel{\cong}{\to} H^i(X_\bullet, a^*L). \tag{5.3}$$

Lemma 5.13. *The isomorphism* (5.3) *holds also in the case when S is a separated DM-stack.*

Proof. Choose an etale hypercovering $Z_\bullet \to S_\bullet$, then we get a bisimplicial algebraic space $\{X_k \times_X Z_l\}_{(k,l)\in\Delta\times\Delta}$. Cohomological descent for the proper surjective topology gives cohomological descent in the k-variable down to Z_l, and the etale hypercovering induces an equivalence of realizations so we have cohomological descent in the l-variable, down to X_k and to S. These allow us to conclude by a spectral sequence argument. $\square$

Lemma 5.14. *Suppose $X_\bullet \to S$ is a proper surjective hypercovering to a separated DM-stack. If L is a local system of sets over $X_\bullet$, then it descends: there exists a local system L_S on S such that $a^*L_S \cong L$.*

Proof. Suppose first that S itself is a separated scheme of finite type over $\mathbb{C}$. Suppose $y \in S$. Let $X_\bullet(y)$ denote the fiber over y, that is $X_k(y)$ is the inverse image of y in X_k. It is nonempty.

Working with local systems and the fundamental group involves only the pieces X_0, X_1, X_2, X_3 of the hypercovering, so for the purposes of the present argument we truncated. This makes it so there are only finitely many k.

The descent data for the local system over the hypercovering imply that L is trivial, i.e. isomorphic to a constant local system, when restricted to $X_\bullet(y)$. To see this, choose a lifting $z \in X_0$ mapping to y, and note that the restriction of L to $X_0 \times_S X_\bullet$ from the second factor, is isomorphic to the pullback of $L|_{X_0}$ from the first factor. Restricting

to $\{z\} \times_S X_\bullet = X_\bullet(y)$ gives the desired trivialization $L|_{X_\bullet(y)} \cong \underline{L_0(z)}$. Note that this trivialization is compatible with the descent data.

Next, note that there exist usual open neighborhoods $X_k(y) \subset W_k \subset X_k$ such that $L|_{W_k}$ has a trivialization compatible with the one constructed previously on $X_k(y)$. To see this, choose an open covering U_k^i on which L is trivialized, then refine it so that any nonempty $U_k^i \cap X_k(y)$ are connected. The trivialization of $L|_{X_k(y)}$ induces a well-defined trivialization of each $L|_{U_k^i}$ for those open sets U_k^i meeting $X_k(y)$; for the other open sets choose arbitrarily. Pass then to relatively compact $V_k^i \subset U_k^i$ which still cover X_k. Now, for any connected component of some $V_k^i \cap V_k^j$ on which the transition isomorphism g_{ij} for L is not the identity, the closure of that connected component misses $X_k(y)$. Taking the complement of the closures of such connected components of $V_k^i \cap V_k^j$ gives a neighborhood W_k of $X_k(y)$ covered by $V_k^i \cap W_k$, such that the transition functions for $L|_{W_k}$ with respect to the covering and the given trivializations, are identities. Patching together gives a global trivialization of $L|_{W_k}$ compatible with the previous one on $X_k(y)$.

The previous trivializations of $L|_{X_k(y)}$ were compatible with the descent data, so possibly reducing the size of W_k we may assume that this is true for our trivializations of $L|_{W_k}$. Properness of the finitely many maps $X_k \to S$ in play, implies that there is a usual open neighborhood $y \in S' \subset S$ such that the inverse image of S' (which we call W'_k) is contained in W_k. The trivialization of $L|_{W'_k}$ then becomes an isomorphism between $L|_{W'_k}$ and the pullback of the constant local system on S', compatible with the descent data. In other words, we have descended $L|_{W'_k}$ to a constant local system on S'. For any point $y \in S$ we obtain such a neighborhood, and the descended local system is unique up to canonical isomorphism (as can be seen by proper surjective descent for sections of local systems). Hence the descended local systems on neighborhoods glue together to give a descent of the local system to L_S on S whose pullback to $X_\bullet$ is L. □

Proposition 5.15. *Suppose $X_\bullet \to S$ is a proper surjective hypercovering to a separated DM-stack. Then this induces a map on topological realisations $|X_\bullet| \xrightarrow{|a|} |S|$ which is a weak homotopy equivalence.*

Proof. The induced map $|a|$ is from (3.1) above. To prove that $|a|$ is a weak equivalence, using Quillen's criterion and cohomological descent (Lemma 5.13 above), it suffices to verify in addition that $|a|$ induces an isomorphism on fundamental groups at any basepoint $x \in X_0$. This is shown by the preceding lemma. □

Remark 5.16. *Suppose $S_\bullet$ is a simplicial projective variety or even a simplicial object in* **DMSt**. *Then there is a map $X_\bullet \to S_\bullet$ which is a weak equivalence for the proper surjective topology, with the X_k being smooth projective varieties. So topologically speaking we don't lose any generality by passing to simplicial smooth projective varieties.*

One can also define the de Rham cohomology of a scheme or stack, using some form of crystalline cohomology, see [101] and [82] for example.

In [104], it is shown that the cohomological descent isomorphism (5.3) also holds for de Rham cohomology. A bisimplicial argument shows that this is also true when S is a separated DM-stack.

To complete the picture of de Rham descent, we note that the analogue of Lemma 5.14 also holds.

Lemma 5.17. *Suppose $X_\bullet \to S$ is a proper surjective hypercovering to a separated DM-stack. Suppose $F_\bullet$ is a compatible system of de Rham local systems with regular singularities on $X_\bullet$, that is F_k is a stratification on the crystalline site of X_k provided with pullback isomorphisms $F_k|_{X_m} \cong F_m$ whenever $m \to k$ in Δ. Then there exists a de Rham local system G on S with $a^*(G) \cong F_\bullet$.*

Proof. The de Rham local systems with regular singularities correspond to local systems by the Riemann-Hilbert correspondence (see [101] about this question for stacks), so this follows from Lemma 5.14. It would be interesting to have a direct algebraic proof, which could apply to irregular de Rham local systems too. $\square$

A further interesting question is the descent of vector bundles along proper surjective hypercoverings. Some understanding of this issue would be helpful in order to obtain an algebraic construction of descent for de Rham local systems.

6. Moduli of local systems on simplicial varieties

Let **ArtSt** denote the 2-category of Artin algebraic stacks of finite type. Suppose $\mathcal{C}$ is a category, and suppose given a 2-functor $\mathbf{F} : \mathcal{C}^o \to$ **ArtSt**. Suppose $X_\bullet$ is a simplicial object of $\mathcal{C}$, that is to say a functor $\Delta^o \to \mathcal{C}$. Then we can define $\mathbf{F}(X_\bullet)$ as the 2-limit of the diagram $\mathbf{F} \circ X_\bullet : \Delta \to \mathbf{ArtSt}$.

Concretely an object $E_\bullet$ of $\mathcal{F}(X_\bullet)$ consists of a collection of objects E_k of $\mathcal{F}(X_k)$ together with isomorphisms $X_\phi^*(E_k) \cong E_m$ whenever $\phi : k \to m$ is a map in Δ inducing $X_\phi : X_m \to X_k$, and these isomorphisms are required to satisfy the obvious compatibility conditions for compositions $k \to m \to l$ and identities.

Lemma 6.1. *The 2-limit $\mathcal{F}(X_\bullet)$ depends only on the start of the simplicial object, in fact it is the 2-limit of the diagram*

$$\mathcal{F}(X_0) \rightrightarrows \mathcal{F}(X_1) \overset{\rightarrow}{\rightrightarrows} \mathcal{F}(X_2).$$

An object $E_\bullet$ of $\mathcal{F}(X_\bullet)$ may also be viewed as just an object E_0 of $\mathcal{F}(X_0)$ together with an isomorphism $\partial_0^* E_0 \cong \partial_1^* E_0$, over X_1, satisfying the cocycle condition when pulled back to X_2.

Proof. The same as for Lemma 5.9 (see the paragraph just before 5.9 for why we don't need to include the degeneracies in the diagram). □

The terminology "simplicial family" will sometimes be useful to describe objects of the form $E_\bullet$. Morphisms in $\mathcal{F}(X_\bullet)$ have corresponding descriptions.

The above construction applies in particular to the category $\mathcal{C}$ of smooth projective varieties. Various functors include:
$X \mapsto \mathcal{M}_B(X, G)$ the moduli stack of representations of $\Pi_1(X)$ in an algebraic group G;
$X \mapsto \mathcal{M}_{DR}(X, G)$ the moduli stack of pairs (P, ∇) where P is a principal G-bundle and ∇ an integrable algebraic connection;
$X \mapsto \mathcal{M}_H(X, G)$ the moduli stack of pairs (P, θ) where P is a principal G-bundle with θ an integrable Higgs field of semiharmonic type (see Definition 7.1);
$X \mapsto \mathcal{M}_{Hod}(X, G)$ the moduli stack of triples (λ, P, ∇) where P is a principal G-bundle and ∇ an integrable algebraic λ-connection of semiharmonic type (which specializes to the preceding two in the cases $\lambda = 1, 0$);
$X \mapsto \mathcal{M}_{DH}(X, G)$ the analytic Deligne-Hitchin moduli stack obtained by glueing two copies of $\mathcal{M}_{Hod}$, noting that here we use $\mathbf{ArtSt}^{\mathrm{an}}$ the 2-category of analytic Artin stacks.

The 2-limit construction gives:

Proposition 6.2. *These functors extend to moduli stacks of various types of local systems denoted $\mathcal{M}_\eta(X_\bullet, G)$ for $\eta = B, DR, H, Hod, DH$, defined for a simplicial object $X_\bullet$ in the category of smooth projective varieties and a linear algebraic group G.*

A more explicit description may also be given. The notion of local system on $X_\bullet$ was discussed above, and indeed it applies to local systems with values in any 1-groupoid. If S is a scheme then $BG(S)$ is the groupoid of G-torsors over S, and $\mathcal{M}_B(X_\bullet, G)(S)$ is the 1-groupoid of local systems with values in $BG(S)$. In other words, an object in $\mathcal{M}_B(X_\bullet, G)(S)$ consists of a locally constant sheaf of G-torsors over S

on each space $|X_k|$ together with isomorphisms between the pullbacks functorial in $k \in \Delta$.

For a base scheme S with a group scheme G/S and function $\lambda : S \to \mathbb{A}^1$, a principal G/S-bundle with λ-connection on $X_\bullet \times S/S$ is a pair $(P_\bullet, \nabla)$ where $P_\bullet$ is a collection of principal $\partial_1^*(G)$-bundles $P(k)$ on $X_k \times S$, with λ-connections ∇ relative to S on each $P(k)$, and compatibility isomorphisms $(P(k), \nabla) \cong (X_\phi)^*(P(m), \nabla)$ whenever $\phi : m \to k$ in Δ, compatible with compositions of ϕ.

If G is a fixed linear algebraic group scheme then apply the previous paragraph with $G \times S/S$. Recall that there are notions of semistability and vanishing of rational Chern classes for principal G-bundles on the projective varieties X_k. The combination of these two conditions is independent of the choice of polarization, and functorial for pullbacks. It will be called "semiharmonic type" in Definition 7.1 below. Say that a principal $G \times S/S$-bundle with λ-connection on $X_\bullet \times S/S$, is of semiharmonic type if its fibers over all $s \in S$ are so.

The moduli stack $\mathcal{M}_{\mathrm{Hod}}(X_\bullet, G)$ is the functor from schemes $S/\mathbb{A}^1$ to 1-groupoids, which to $S \xrightarrow{\lambda} \mathbb{A}^1$ associates the 1-groupoid of principal $G \times S/S$-bundles with λ-connection of semiharmonic type. Specializing to $\lambda = 0$ and $\lambda = 1$ yields the Hitchin and de Rham moduli stacks respectively.

Equivalences between moduli stacks which are natural in the variable X translate in the simplicial setting to equivalences. This gives the Riemann-Hilbert equivalence

$$\mathcal{M}_{DR}(X_\bullet, G)^{\mathrm{an}} \cong \mathcal{M}_B(X_\bullet, G)^{\mathrm{an}}$$

which in turn allows us to construct the analytic moduli stack

$$\mathcal{M}_{DH}(X_\bullet, G) \to \mathbb{P}^1$$

by the Deligne-Hitchin glueing [97].

For $G = GL(n)$ letting n vary we obtain the categories of local systems of vector spaces, or of bundles with λ-connection. In this case we may consider all morphisms not necessarily isomorphisms, and the same considerations as above apply.

Say that $X_\bullet$ is connected if its topological realization is connected. If $x : Spec(\mathbb{C}) \to X_0$ is a basepoint, we obtain a map of Artin stacks

$$x^* : \mathcal{M}_\eta(X_\bullet, G) \to BG.$$

Let $R_\eta(X_\bullet, x, G)$ denote the fiber of x^* over the standard basepoint $0 \in BG$. When $X_\bullet$ is connected, the correspondences between G-torsors and bundles with integrable connection, or Higgs bundles of

semiharmonic type, imply that all points $R_\eta(X_\bullet, x, G)$ have trivial stabilizers.

As in Proposition 2.3, one should choose multiple base points in order to express $R_\eta(X_\bullet, x, G)$ in terms of the representation spaces of the components X_k. Choose a nonempty simplicial set $\mathbf{x}_\bullet$, finite at each level, with a map $\mathbf{x}_\bullet \to X_\bullet$. An η-local system with coefficients in G, such as a flat G-torsor for $\eta = B$ or a principal G-Higgs bundle for $\eta = H$, restricts to a simplicial family of vector spaces over $\mathbf{x}_\bullet$. In all cases this corresponds to a flat G-torsor on the realization $|\mathbf{x}_\bullet|$. A framing is a trivialization of this flat G-torsor. If we assume that $|\mathbf{x}_\bullet|$ is homotopically discrete and choose a set of points mapping isomorphically to its π_0, then a framing is the same thing as a framing of the collection of fibers of our torsor over the given points.

Let $R_\eta(X_\bullet, \mathbf{x}_\bullet, G)$ denote the moduli stack of η-local systems on $X_\bullet$ with coefficients in G, framed along $\mathbf{x}_\bullet$.

Proposition 6.3. *With the above notations, the stack $R_\eta(X_\bullet, \mathbf{x}_\bullet, G)$ is the 2-limit of the diagram $k \mapsto R_\eta(X_k, \mathbf{x}_k, G)$.*

If furthermore the finite sets $\mathbf{x}_k$ meet each component of X_k for $k = 0, 1, 2$, then $R_\eta(X_\bullet, \mathbf{x}_\bullet, G)$ is the equalizer of the two maps between the pieces for $k = 0, 1$:

$$R_\eta(X_\bullet, \mathbf{x}_\bullet, G) \to R_\eta(X_0, \mathbf{x}_0, G) \rightrightarrows R_\eta(X_1, \mathbf{x}_1, G). \tag{6.1}$$

In particular, it is a quasiprojective scheme.

Proof. The first part is formal. The 2-limit only depends on the first three terms. If the simplicial basepoint meets all components of the X_k then the terms $R_\eta(X_k, \mathbf{x}_k, G)$ are quasiprojective schemes [96], so the 2-limit is just the equalizer. □

As discussed in Proposition 2.3 and Example 2.4 for local systems (that is $\eta = B$), the condition that the basepoint meets the components of X_2 is necessary for this to be true even though it doesn't then enter into the formula.

Suppose the set of basepoints is smaller, for example a single x. In the Betti case $R_\eta(X_\bullet, x, G)$ is a quasiprojective scheme. In fact it is just the usual affine scheme of representations of $\pi_1(|X_\bullet|, x)$. By the Riemann-Hilbert correspondence, separability follows for the de Rham case $\eta = DR$. For Higgs bundles, a geometrical argument seems to be needed and will be formulated in the next theorem.

Lemma 6.4. *Suppose Y is a smooth projective variety and P, Q are principal G-bundles with λ-connection on $Y \times S/S$ for a quasiprojective base scheme S. Then the functor which to $S' \to S$ associates*

the set of isomorphisms between $P|_{Y\times S}$ and $Q|_{Y\times S}$ is represented by a quasiprojective S-scheme $\mathrm{Iso}_{Y\times S/S}(P,Q)$ affine over S.

Proof. Embedd $G\subset GL(n)$. Morphisms between linear bundles with λ-connection are representable by vector schemes. Isomorphisms are then parametrized by pairs of morphisms going both ways whose composition is the identity, and the condition that the isomorphism respect the reduction of structure group to G is a closed condition. $\square$

Theorem 6.5. *Suppose $X_\bullet$ is a connected simplicial smooth projective variety, with a nonempty simplicial basepoint $\mathbf{x}_\bullet \to X_\bullet$. Then $R_\eta(X_\bullet, \mathbf{x}_\bullet, G)$ is a quasiprojective (in particular separated) scheme.*

If $z\in X_k$, let $\mathbf{x}'_\bullet = \mathbf{x}_\bullet \sqcup \langle z\rangle$. Then G acts freely on $R_\eta(X_\bullet, \mathbf{x}'_\bullet, G)$ by change of framing at z, and the quotient is $R_\eta(X_\bullet, \mathbf{x}_\bullet, G)$.

Proof. Notice that the second statement follows from the first, because $R_\eta(X_\bullet, \mathbf{x}'_\bullet, G)$ is the G-bundle of frames of the universal bundle over $R_\eta(X_\bullet, \mathbf{x}_\bullet, G)$.

No argument is needed for $\eta = B$, so we will be treating G-principal λ-connections. The condition of semiharmonic type, i.e. semistability and vanishing of Chern classes, is assumed everywhere.

For any simplicial basepoint $\mathbf{x}_\bullet$, even if it doesn't meet all components of the X_k, let $\mathcal{R}_\eta(X_{\leq 1}, \mathbf{x}_{\leq 1}, G)$ denote the moduli stack of η-local systems on the 1-skeleton of $X_\bullet$, trivialized over the 1-skeleton of $\mathbf{x}_\bullet$. In general it might be a stack, as occurs when $\mathbf{x}_\bullet$ is empty for example.

It parametrizes degeneracy-compatible descent data on $X_1 \rightrightarrows X_0$, that is to say pairs (P,ϕ) where P is a G-bundle with λ-connection of semiharmonic type on X_0, and $\phi : \partial_0^*(P) \cong \partial_1^*(P)$ such that $s_0^*(\phi) = 1$ where $s_0 : X_0 \to X_1$ is the degeneracy. This condition needs to be included here or else one would have to talk about $(X_{\leq 1})_2$ which is nonempty but has only degenerate pieces.

If $\mathbf{x}_\bullet$ meets all connected components of the X_k then compatibility with the degeneracy is automatic, and $\mathcal{R}_\eta(X_{\leq 1}, \mathbf{x}_{\leq 1}, G)$ is the equalizer (6.1) occuring in Proposition 6.3. In this case it is a quasiprojective scheme rather than a stack.

For components of the simplicial basepoint of the form $\langle y\rangle \cong h([2])$ for $y\in X_2$, the 1-skeleton is not contractible: rather it is the boundary triangle of the 2-simplex. This case is what leads to new equations for the representation varieties when we add in points to the simplicial basepoint, so it is worth looking at more closely. Let $T_\bullet := h([2])$ denote the contractible simplicial 2-simplex. Its boundary or 1-skeleton $T_{\leq 1}$ is a triangle. Let $t_0 \in T_0$ denote the 0-th vertex.

The inclusion of the boundary triangle into the 2-simplex induces the map

$$\ast = \mathcal{R}_\eta(T_\bullet, (t_0)_{\leq 1}, G) \to \mathcal{R}_\eta(T_{\leq 1}, (t_0)_{\leq 1}, G) = G \tag{6.2}$$

which is inclusion of the identity element as a point in G. This may be seen directly from the configuration of three points in T_0 corresponding to vertices of the triangle and six points of T_1, three degenerate ones located at the vertices and three corresponding to the nondegenerate edges. A principal G-bundle over this configuration together with its face and degeneracy maps corresponds to a flat G-bundle on the boundary of the triangle. When the trivialization at t_0 is included, it corresponds to a monodromy element in G, and it extends to all of T if and only if the monodromy element is trivial.

Continue with the proof of the theorem. Suppose given a simplicial basepoint of the form $\mathbf{x} = \langle x^1 \rangle \sqcup \cdots \sqcup \langle x^r \rangle$ with $x^1 \in X_0$ and $x^i \in X_{k^i}$ for $k^i \in \{0, 1, 2\}$. Let $G'(\mathbf{x}_\bullet) := \prod_{j=2}^r G$ with the j-th term acting by change of framing over the 0-th vertex of x^j. Thus $G'(\mathbf{x}_\bullet)$ acts on $\mathcal{R}_\eta(X_{\leq 1}, \mathbf{x}_{\leq 1}, G)$.

If $\mathbf{x}_\bullet$ meets all connected components of X_0, X_1 and X_2 then Proposition 6.3 tells us that $R(X_\bullet, \mathbf{x}_\bullet, G) = \mathcal{R}_\eta(X_{\leq 1}, \mathbf{x}_{\leq 1}, G)$ hence

$$R(X_\bullet, x^1, G) = \mathcal{R}_\eta(X_{\leq 1}, \mathbf{x}_{\leq 1}, G) /\!/ G'(\mathbf{x}_\bullet).$$

The goal is to show that this is quasiprojective.

Start with a simpler choice of simplicial basepoint. Order the connected components of X_0 as $X_0^1, \ldots, X_0^a$, such that for any $2 \leq i \leq a$ there exists a connected component X_1^i of X_1 with $\partial_0(X_1^i) \subset X_0^j$ for $j < i$ and $\partial_1(X_1^i) \subset X_0^i$. Choose $y^1 \in X_0^1$, and for $2 \leq i \leq a$ choose $y^i \in X_1^i$. Then set

$$\mathbf{y}_\bullet = \langle y^1 \rangle \sqcup \cdots \sqcup \langle y^a \rangle.$$

Consider first the equalizer

$$\mathcal{R}' \to R(X_0, \mathbf{y}_0, G) \rightrightarrows R(\coprod_{i=2}^a X_1^i, \mathbf{y}_1, G).$$

The quotient $\mathcal{R}' /\!/ G(\mathbf{y}_\bullet)$ is a moduli stack parametrizing a-tuples of G-principal λ-connections P^i on the X_0^i, with a choice of framing for P^1 over x^1, together with choices of isomorphisms between the restrictions $\partial_0^*(P^{i-1})$ and $\partial_1^*(P^{i-1})$ over X_1^i for $i = 2, \ldots, a$.

Let V_k denote the moduli stack of k-uples $(P^1, \ldots, P^k)$ with isomorphisms as above. We prove by induction on k that it is a quasiprojective scheme, starting with $k = 1$ which is the case of principal λ-connections over the smooth projective variety X_0^1 [96].

There is a universal object over $U_k \times X_0^1 \times \cdots \times X_0^a$, in particular its restriction to the next basepoint $\partial_0(y^{k+1})$ is a principal G-bundle over U_k. The representation variety $R(X_0^{k+1}, \partial_1(y^{k+1}), G)$ has a G-action, so we can twist it to get a fibration $V_{k+1} \rightarrow U_k$ with fiber $R(X_0^{k+1}, \partial_1(y^{k+1}), G)$. Similarly, twisting gives a fibration $W_{k+1} \rightarrow U_k$ with fiber $R(X_1^{k+1}, y^{k+1}, G)$. Restriction of the universal bundle is a section $U_k \rightarrow W_{k+1}$, and restriction from X_0^{k+1} is a morphism $V_k \rightarrow W_{k+1}$. Specifying a $k+1$-tuple $(P^1, \ldots, P^{k+1})$ is equivalent to specifying a point in V_{k+1} whose restriction is the same as that of P^k. In other words, the next moduli space is the fiber product

$$U_{k+1} = V_{k+1} \times_{W_{k+1}} U_k.$$

Hence U_{k+1} is a quasiprojective scheme. This completes the inductive step. At $k = a$, this shows that

$$U_a = \mathcal{R}' /\!/ G(\mathbf{y}_\bullet)$$

is quasiprojective.

Now $\mathcal{R}_\eta(X_{\leq 1}, \mathbf{y}_{\leq 1}, G) /\!/ G'(\mathbf{y}_\bullet)$ is affine over $\mathcal{R}' /\!/ G(\mathbf{y}_\bullet)$ parametrizing isomorphisms between the restrictions ∂_0^* and ∂_1^* of the universal object, to the other components of X_1. Lemma 6.4 applied to the union of other components, gives that $\mathcal{R}_\eta(X_{\leq 1}, \mathbf{y}_{\leq 1}, G) /\!/ G'(\mathbf{y}_\bullet)$ is quasiprojective.

To finish, proceed by induction starting from $\mathbf{y}$ and successively adding points until we get to a simplicial basepoint meeting all the required components. It suffices analyze what happens when we pass from $\mathbf{x}_\bullet$ to $\mathbf{x}_\bullet \sqcup \langle z \rangle$ for $z \in X_k$, $k = 0, 1, 2$. For in X_0 or X_1, the moduli problem solved by $\mathcal{R}_\eta(X_{\leq 1}, \mathbf{y}_{\leq 1} \sqcup \langle z \rangle_{\leq 1}, G) /\!/ G'(\mathbf{y}_\bullet \sqcup \langle z \rangle)$ is the same as that solved by $\mathcal{R}_\eta(X_{\leq 1}, \mathbf{y}_{\leq 1}, G) /\!/ G'(\mathbf{y}_\bullet)$, plus a choice of framing over z, but also modulo the action of an extra copy of G on this choice of framing. Therefore

$$\mathcal{R}_\eta(X_{\leq 1}, \mathbf{y}_{\leq 1} \sqcup \langle z \rangle_{\leq 1}, G) /\!/ G'(\mathbf{y}_\bullet \sqcup \langle z \rangle) \cong \mathcal{R}_\eta(X_{\leq 1}, \mathbf{y}_{\leq 1}, G) /\!/ G'(\mathbf{y}_\bullet)$$

and quasiprojectivity for $\mathbf{y}_\bullet$ implies quasiprojecxtivity for $\mathbf{y}_\bullet \sqcup \langle z \rangle$.

Consider therefore the case $z \in X_2$. Then $\langle z \rangle$ is the 2-simplex T considered above. A descent datum on $X_{\leq 1}$ restricts to one over $T_{\leq 1}$, hence to a monodromy element as discussed after equation (6.2) above. The moduli problem solved by $\mathcal{R}_\eta(X_{\leq 1}, \mathbf{y}_{\leq 1} \sqcup \langle z \rangle_{\leq 1}, G) /\!/ G'(\mathbf{y}_\bullet \sqcup \langle z \rangle)$ is the moduli problem for $U := \mathcal{R}_\eta(X_{\leq 1}, \mathbf{y}_{\leq 1}, G) /\!/ G'(\mathbf{y}_\bullet)$, plus a trivialization of this G-bundle on the boundary of the triangle $T_{\leq 1}$, modulo choice of framing at one point.

Over $U \times X_{\leq 1}$ there is a universal object which restricts to a G-bundle on $U \times T_{\leq 1}/U$. The condition that the monodromy be trivial is a closed condition over U. To prove this it suffices to do it etale-locally,

but then we can assume that there is a trivialization of the restriction to one vertex; the monodromy becomes a function to G such that the inverse image of $\{1_G\}$ (see (6.2)) is the required closed subset. From all of this we conclude that

$$\mathcal{R}_\eta(X_{\le 1}, \mathbf{y}_{\le 1} \sqcup \langle z\rangle_{\le 1}, G)/\!/G'(\mathbf{y}_\bullet \sqcup \langle z\rangle) \subset \mathcal{R}_\eta(X_{\le 1}, \mathbf{y}_{\le 1}, G)/\!/G'(\mathbf{y}_\bullet)$$

is a closed subscheme. Again, quasiprojectivity on the right implies it on the left. This completes the induction step.

By induction, we can go to the case of a simplicial basepoint $\mathbf{x}_\bullet$ meeting all components of X_k for $k = 0, 1, 2$. We have shown that $\mathcal{R}_\eta(X_{\le 1}, \mathbf{x}_{\le 1}, G)/\!/G'(\mathbf{x}_\bullet)$ is a quasiprojective scheme. However, it is equal to R in this case by Proposition 6.3, which completes the proof that

$$R(X_\bullet, x^1, G) = R(X_\bullet, \mathbf{x}_\bullet, G)/\!/G'(\mathbf{x}_\bullet)$$

is quasiprojective. This finishes the proof of the theorem in case of a single basepoint. For any nonempty simplicial basepoint, going back in the other direction corresponds to looking at frame bundles over $R(X_\bullet, x^1, G)$, which are quasiprojective too. $\square$

This proof shows how the components of X_2 lead to additional equations for the representation scheme, via the monodromy elements over triangles $T_{\le 1}$. In case of a simplicial scheme $X_\bullet$ such that each X_k is simply connected, the fundamental group is the same as that of the simplicial set $k \mapsto \pi_0(X_k)$ and the above procedure shows how the elements of $\pi_0(X_2)$ act as relations.

Corollary 6.6. *Suppose $X_\bullet$ is connected with each X_k being a smooth projective variety. Choose a basepoint $x \in X_0$. Then $R_\eta(X_\bullet, x, G)$ is a quasiprojective scheme for $\eta = B, DR, H, Hod$, an analytic space for $\eta = DH$. The group G acts on it and the quotient stack is $\mathcal{M}_\eta(X_\bullet, G)$. For the cases $\eta = H, Hod, DH$ there is an action of $\mathbb{G}_{\mathrm{m}}$ on both the representation scheme R and the quotient stack $\mathcal{M}$.*

Proof. Apply the previous theorem with the nonempty basepoint $\langle x\rangle$. The group actions are obtained from the universal property. $\square$

For an explicit description of the representation scheme, let $y^1 = x \in X_0$ be the first basepoint. Choose $y^j \in \mathbf{x}_{m(j)}$ for $j = 2, \ldots, r$ such that the collection meets all components of X_0, X_1 and X_2. Let $\mathbf{y}_\bullet = \coprod_{j=1}^r \langle y^j\rangle$ be the corresponding simplicial basepoint. Choose representatives $x^j \in \langle y^j\rangle_0$.

Corollary 6.7. *With these notations, Proposition 6.3 allows us to calculate $R_\eta(X_\bullet, \mathbf{y}_\bullet, G)$. Then, applying Theorem 6.5 recursively, the*

group $\prod_{j=2}^r G$ *acts freely on* $R_\eta(X_\bullet, \mathbf{y}_\bullet, G)$ *by change of framings at the points* x^j, *and the quotient is* $R_\eta(X_\bullet, x, G)$. *It extends to an action of the group* $\prod_{j=1}^r G$ *with*

$$R_\eta(X_\bullet, \mathbf{y}_\bullet, G)/\!/(G \times \prod_{j=1}^{r} G) \cong R_\eta(X_\bullet, x, G)/\!/G \cong \mathcal{M}_\eta(X_\bullet, G).$$

Choosing only basepoints in X_0 gives a slightly different description refering directly to Lemma 6.4.

Corollary 6.8. *Choose basepoints* $y^j \in X_0$ *meeting all the connected components of* X_0, *and let* $\mathbf{y}_\bullet = \coprod_{j=1}^r \langle y^j \rangle$. *Then*

$$\mathcal{R}_\eta(X_{\leq 1}, \mathbf{y}_{\leq 1}, G) \to R_\eta(X_0, \{y^j\}, G)$$

is an affine map parametrizing G*-principal* λ*-connections* P *on* X_0, *framed at the* y^j, *together with isomorphisms* $\phi : \partial_0^*(P) \cong \partial_1^*(P)$ *on* X_1 *compatible with the degeneracies (or equivalently, with the framings on* X_1*). Furthermore*

$$R_\eta(X_\bullet, \mathbf{y}_\bullet, G) \subset \mathcal{R}_\eta(X_{\leq 1}, \mathbf{y}_{\leq 1}, G)$$

is a closed subvariety parametrizing the (P, ϕ) *such that* ϕ *satisfies the cocycle condition on* X_2.

Turn now to the study of the universal categorical quotients of the moduli stacks, going back to [73] and more particularly [65] for the moduli of representations. Consider first the abstract setup of an algebraic stack $\mathcal{M}$, similarly to the work of Iwanari [52]. A morphism $\mathcal{M} \to M$ is a universal categorical quotient in the category of schemes if M is a scheme, and if for any schemes Y and Z with a map $Z \to M$, a map

$$\mathcal{M} \times_M Z \to Y$$

factors through a unique map $Z \to Y$. A universal categorical quotient is obviously unique.

In our situation, the moduli stack is a quotient stack $\mathcal{M} = R/\!/G$ with R a quasiprojective scheme. In this case, Seshadri defines the notion of good quotient [92] and notes that Mumford's construction of the quotient for the set of semistable points [73] is good. A good quotient is separated and quasiprojective, and the points correspond to closed orbits of the G-action.

Lemma 6.9. *Suppose* V *is a quasiprojective scheme with* G *action, such that all points are semistable with respect to a linearized line bundle* L. *Suppose* $\varphi : F \to G$ *is a* G*-equivariant affine map. Then all*

points of F are semistable for $\varphi^*(L)$, so there is a good quotient F/G too.

Proof. If $z \in F$ then $\varphi(z)$ is semistable by hypothesis. This means that there is a section $f \in H^0(V, L^{\otimes n})^G$ such that $V_{f\neq 0}$ is an affine neighborhood of $\varphi(z)$. It pulls back to a G-invariant section on F and $F_{\varphi^*(f)\neq 0} = \varphi^{-1}(V_{f\neq 0})$ is affine. Thus z is semistable. □

Theorem 6.10. *Suppose $Z_\bullet$ is a simplicial scheme with split degeneracies such that the Z_k are smooth projective varieties. Suppose $Z_\bullet$ is connected with a basepoint z. Then for $\eta = B, DR, H, Hod$ there is a linearized line bundle such that all points of $R_\eta(Z_\bullet, z, G)$ are semistable. Therefore $\mathcal{M}_\eta(Z_\bullet, G)$ admits a universal categorical quotient which is a good quotient*

$$M_\eta(Z_\bullet, G) = R_\eta(Z_\bullet, z, G)/G.$$

Proof. Choose points $z = y^1, \ldots, y^b$ in all the connected components of Z_0 and let $\mathbf{y}_\bullet$ be the corresponding simplicial basepoint of $Z_\bullet$. Then the action of G^b on $R_\eta(Z_0, \{y^j\}, G)$ by change of framings, linearizes a line bundle L_0 for which all points are semistable. By Corollary 6.8 the map

$$R_\eta(Z_\bullet, \mathbf{y}_\bullet, G) \to R_\eta(Z_0, \{y^j\}, G)$$

is a G^b-equivariant affine map. Therefore L_0 pulls back to a G^b-linearized line bundle $\widetilde{L}$ on $R_\eta(Z_\bullet, \mathbf{y}_\bullet, G)$ for which all points are semistable and there exists a good quotient, as pointed out in Lemma 6.9. The quotient map factors through a good quotient by G^{b-1} first, then the quotient by G:

$$R_\eta(Z_\bullet, \mathbf{y}_\bullet, G) \to R_\eta(Z_\bullet, y^1, G) \to M_\eta(Z_\bullet, G). \tag{6.3}$$

In the middle is a quasiprojective scheme representing the corresponding functor, by Theorem 6.5. The line bundle $\widetilde{L}$ descends to a G-linearized bundle L on $R_\eta(Z_\bullet, y^1, G)$, which is the pullback of a bundle on the good G^b-quotient $M = M_\eta(Z_\bullet, G)$. All of the maps in (6.3) are affine maps since the middle variety is covered by the affine G^b-quotients of the inverse images of affine sets defined by sections of the line bundle on M. It follows that all points of $R_\eta(Z_\bullet, y^1, G)$ are semistable, and M is a good quotient of $R_\eta(Z_\bullet,^1, G)$ by the action of G. □

Corollary 6.11. *The universal categorical quotients $\mathcal{M}_{\mathrm{Hod}}(X_\bullet, G) \to M_{\mathrm{Hod}}(X_\bullet, G)$ glue together to give a separated analytic universal categorical quotient*

$$\mathcal{M}_{\mathrm{DH}}(X_\bullet, G) \to M_{\mathrm{DH}}(X_\bullet, G)$$

which is the Deligne-Hitchin twistor space for representations of $\pi_1(X_\bullet)$ *in* G.

One can interpret these things as a kind of *weight filtration*. Suppose $X_\bullet$ is a simplicial scheme such that each X_k is a smooth projective variety. The morphisms

$$\mathcal{M}_\eta(X_\bullet, G) \to \mathcal{M}_\eta(X_0, G)$$

$$M_\eta(X_\bullet, G) \to M_\eta(X_0, G)$$

and, for $x \in X_0$ a basepoint,

$$R_\eta(X_\bullet, x, G) \to R_\eta(X_0, x, G),$$

induce equivalence relations on the left hand sides. Define the weight filtration to be the equivalence relation[1]

$$W\mathcal{M}_\eta(X_\bullet, G) := \mathcal{M}_\eta(X_\bullet, G) \times_{\mathcal{M}_\eta(X_0,G)} \mathcal{M}_\eta(X_\bullet, G),$$

and similarly for WM_η and WR_η. The WM_η and WR_η are equivalence relations on M_η and R_η respectively. Because of the stackiness, $W\mathcal{M}_\eta$ will in general have a structure of groupoid in the category of stacks. However, the arguments given above show that the map $\mathcal{M}_\eta(X_\bullet, G) \to \mathcal{M}_\eta(X_0, G)$ is representable and affine.

7. Hodge and harmonic theory

Classical results and techniques from Hodge theory apply also to Deligne-Mumford stacks, see [103] [68] for example, and more generally to simplicial manifolds as in [39] [55]. Similarly, nonabelian harmonic theory for local systems applies to a simplicial smooth projective variety, with a few modifications, by working on each level. Many proofs in this section will be shortened or left to the reader.

Suppose $X_\bullet$ is a simplicial smooth projective variety. A simplicial Higgs bundle $(E_\bullet, \theta)$ is a collection of Higgs bundles E_k of rank n on X_k, together with compatibility isomorphisms for each $k \to m$ in Δ in the same way as for local systems.

If G is a linear algebraic group, a principal G-Higgs bundles $(P_\bullet, \theta)$ on $X_\bullet$ is a simplicial family of principal G-Higgs bundles on the X_k. The preceding definition is recovered for $G = GL(n)$. Make the corresponding definitions for λ-connections over $\lambda \in \mathbb{A}^1$.

[1] It might be interesting to use the derived fiber product here instead, but that would go beyond our present scope.

Definition 7.1. *A principal G-Higgs bundle (P, θ) on a smooth projective variety X is of semiharmonic type if it is semistable with Chern clases vanishing in rational cohomology. This condition is independant of the choice of Kähler class.*

A simplicial principal G-Higgs bundle $(P_\bullet, \theta)$ over a simplicial smooth projective variety $X_\bullet$ is said to be of semiharmonic type if each (P_k, θ) is of semiharmonic type on X_k.

Given a principal G-bundle with λ-connection $(P_\bullet, \nabla)$, say that it is of semiharmonic type if it satisfies the previous definition when $\lambda = 0$; the condition is automatically true for $\lambda \neq 0$ since we consider only the compact case here.

Applying the equivalence of categories from [95] level by level gives a simplicial version.

Proposition 7.2. *Suppose $X_\bullet$ is a connected simplicial smooth projective variety. There is an equivalence of tannakian categories between the category of simplicial Higgs bundles of semiharmonic type on $X_\bullet$ and the category of local systems. This equivalence is compatible with pullback along morphisms of simplicial varieties, in particular it preserves the fiber functors of restriction to a basepoint. For any linear group G this induces an equivalence between the categories of simplicial principal G-Higgs bundles of semiharmonic type on $X_\bullet$ and the category of G-torsors over $|X_\bullet|$.*

Say G is reductive. A principal Higgs bundle of semiharmonic type will be called polarizable if each (P_k, θ) admits a harmonic reduction of structure group to the maximal compact of G, or equivalently if it is polystable. Say that $(P_\bullet, \theta)$ is *strongly polarizable* if there is a simplicial family of harmonic reductions of structure group h_k compatible under the transition maps for $k \to m$ in Δ. Similarly, say that a local system is strongly polarizable if there exists a compatible collection of harmonic metrics h_k on L_k. We use this terminology interchangeably for the corresponding local system L on $|X_\bullet|$.

The equivalence of categories of Proposition 7.2 preserves the conditions of polarizability and strong polarizability. For polarizable objects the equivalence can be expressed in terms of harmonic bundles on $X_\bullet$, in other words simplicial families denoted $\mathcal{E}_\bullet$ of harmonic bundles $(\mathcal{E}_k, \partial, \overline{\partial}, \theta, \overline{\theta})$ on X_k, together with pullback isomorphisms compatible with cohomology.

The category of harmonic bundles on a simplicial scheme $X_\bullet$ maps by an equivalence of category to the subcategory of polarizable local systems $L_\bullet$ on $X_\bullet$, i.e. ones such that each L_k is semisimple on X_k. It

also maps by an equivalence of categories to the category of termwise polystable λ-connections, for any $\lambda \in \mathbb{A}^1$. Among other things, these functors with the same formulae as in the usual smooth projective case, provide us with a collection of prefered sections of the family of analytic moduli stacks $\mathcal{M}_{\mathrm{DH}}(X_\bullet, G) \to \mathbb{P}^1$, and their images which are sections of the family of moduli spaces $M_{\mathrm{DH}}(X_\bullet, G) \to \mathbb{P}^1$.

Proposition 7.3. *The category of strongly polarizable local systems is tannakian and semisimple. Restriction to any basepoint $x \in |X_\bullet|$ provides a fiber functor, and the corresponding affine algebraic group $\varpi_1^{\mathbf{SP}}(X_\bullet, x)$ is reductive. The monodromy representation of a strongly polarizable local system is semisimple.*

Proof. Suppose $(L_\bullet, h_\bullet)$ is a strongly polarized local system on $X_\bullet$. If $U_\bullet \subset L_\bullet$ is a sub-local system then the simplicial family $V : k \mapsto U_k^\perp$ of orthogonal complements with respect to the h_k forms a complement, $L_\bullet = U_\bullet \oplus V_\bullet$. □

The following example shows that semisimplicity doesn't necessarily hold for local systems which are only polarizable; and furthermore that semisimple local systems are not necessarily strongly polarizable.

Example 7.4. *Suppose $X_\bullet$ is a simplicial smooth projective variety such that each X_k is simply connected. Then every local system L on $|X_\bullet|$ is polarizable, but a local system is strongly polarizable if and only if it is unitary.*

In fact, semisimplicity doesn't necessarily imply polarizability, either.

Example 7.5. *Let $X_\bullet$ be the simplicial resolution of a nodal curve, with X_0 the normalization of genus $g > 1$. Then there are local systems which are not semisimple on X_0, but where the additional monodromy transformation at the node makes the full monodromy representation semisimple.*

On the other hand, for hypercoverings of normal DM-stacks,

$$\pi_1(X_0, x) \to \pi_1(|X_\bullet|, x)$$

has image of finite index and polarizability, semisimplicity and strong polarizability are the same—see Lemma 8.2 and Theorem 8.4 below.

If $X_\bullet$ is a simplicial smooth scheme, a variation of Hodge structure $V_\bullet$ over $X_\bullet$ consists of specifying a variation of Hodge structure V_k on each X_k, together with functoriality isomorphisms $\phi^*(V_k) \cong V_m$ whenever $\phi : [k] \to [m]$ is a map in Δ, satisfying the usual compatibility condition. We say that $V_\bullet$ is polarizable if each V_k is polarizable. We

say that $V_\bullet$ is strongly polarizable if there exist polarizations h_k on each V_k which are compatible with the functoriality isomorphisms.

Lemma 7.6. *Suppose $X_\bullet$ is a simplicial smooth projective variety. Suppose $L_\bullet$ is a polarizable local system on $X_\bullet$ corresponding to the Higgs bundle $(E_\bullet, \theta)$. Then a structure of polarizable VHS on $L_\bullet$ is exactly given by a trivialization of the $\mathbb{C}^*$ action $\varphi_t : (E_\bullet, \theta) \cong (E_\bullet, t\theta)$. A strongly polarizable local system which is a fixed point, corresponds to a strongly polarizable variation of Hodge structure.*

Remark 7.7. *Suppose $V_\bullet$ is a strongly polarizable VHS. Then the monodromy group of the underlying representation of $\pi_1(|X_\bullet|)$ is contained in some $U(p,q)$. However, this is not necessarily the case for a polarizable VHS which is not strongly polarizable.*

Conjecture 7.8. *For a strongly polarizable variation of Hodge structure, the real Zariski closure of the image of $\pi_1(|X_\bullet|, x)$ in $GL(L(x))$ is a group of Hodge type.*

If $X_\bullet$ is a simplicial variety whose components are simply connected, then any local system is trivial on each X_k, in particular setting $V_k^{0,0} := L_k$ gives a polarizable VHS, which will not however usually be strongly polarizable. So in general the existence of a polarizable VHS doesn't lead to restrictions on the representation or the fundamental group. For that, one requires the finite index condition 8.1, as will be discussed in the next section on normal DM-stacks. That condition implies Conjecture 7.8.

The following lemma shows that lack of strong polarizability is an obstruction to extending a local system to a smooth ambient variety.

Lemma 7.9. *Suppose $X_\bullet \to Z$ is a morphism from a simplicial smooth projective variety, to a smooth quasiprojective variety Z; for example when $X_\bullet$ is a proper surjective hypercovering of a closed subscheme of Z. If $L_\bullet$ is a semisimple local system on $X_\bullet$ which is the pullback of a local system on Z, then it is strongly polarizable.*

Proof. If $L_\bullet$ is the pullback of a local system L_Z, then it is also the pullback of the associated-graded of the Jordan-Hölder series for L_Z, so we may assume L_Z semisimple; it then has a harmonic metric [71] which restricts to a strong polarization of $L_\bullet$. □

There are other obstructions of a similar nature. Suppose $X_\bullet$ is a simplicial smooth projective variety, and $x \in X_1$ is a point. It has two images $\partial_0 x, \partial_1 x \in X_0$. If $(E_\bullet, \theta)$ is a Higgs bundle on $X_\bullet$ then

$$E_0(\partial_0 x) \cong E_1(x) \cong E_0(\partial_1 x)$$

so the Higgs field θ on E_0 over X_0 provide separately commutative actions of the two tangent spaces $T_{\partial_0 x}X_0$ and $T_{\partial_1 x}X_0$ on $E_1(x)$. Say that $(E_\bullet, \theta)$ satisfies the commutativity obstruction if these two actions commute with each other, in other words $\theta(\partial_0 x)(v_0)$ and $\theta(\partial_1 x)(v_1)$ commute as endomorphisms of $E_1(x)$ whenever $v_0 \in T_{\partial_0 x}X_0$ and $v_1 \in T_{\partial_1 x}X_0$. The following lemma identifies this as another obstruction to extending a local system to a smooth ambient variety.

Lemma 7.10. *Suppose $f : X_\bullet \to Z$ is a morphism from a simplicial smooth projective variety to a smooth quasiprojective variety Z, and $L_\bullet$ is a local system on $X_\bullet$ corresponding to a Higgs bundle $(E_\bullet, \theta)$. If $L_\bullet$ is the pullback of a local system on Z then $(E_\bullet, \theta)$ satisfies the commutativity obstruction at each $x \in X_1$.*

Proof. If $L_\bullet$ is a pullback from Z, then $(E_\bullet, \theta)$ is the pullback of a Higgs bundle (F, φ) on Z (this works even if Z is only quasiprojective by [71]). For any $x \in X_1$, the actions of both tangent spaces $T_{\partial_0 x}X_0$ and $T_{\partial_1 x}X_0$ factor through the action of $T_{f(x)}Z$ on $E_1(x) \cong F(f(x))$ given by $\varphi_{f(x)}$. $\square$

To give a concrete example, suppose X is a nodal curve embedded in a smooth variety Z. A local system L on X restricts to a local system corresponding to a Higgs bundle (E_0, θ) on the normalization $X_0 = \tilde{X}$. At each node $x \in X$ we obtain two endomorphisms of $L(x)$ given by the Higgs field θ applied to the tangent vectors along the two branches going through x. The commutativity obstruction says that these should commute, as will be the case if the Higgs bundle is a pullback from Z.

Look now at the local structure of the space of representations. If $\mathcal{G}$ is an algebraic stack and $p \in \mathcal{G}(\mathbb{C})$ is a closed point, the tangent space $T_p\mathcal{G}$ is defined as the set of pairs (f, e) where

$$f : \mathrm{Spec}\mathbb{C}[\varepsilon]/\varepsilon^2 \to \mathcal{G}$$

and $e : f|_{\mathrm{Spec}\mathbb{C}} \cong p$ is an isomorphism of points, up to natural equivalences of the f respecting the isomorphisms e. If $\mathcal{G}$ is a moduli stack then the tangent space is usually known as the deformation space: a point consists of an infinitesimal deformation with isomorphism between the central fiber and the original object in question.

Suppose $X_\bullet$ is a connected simplicial smooth projective variety, with basepoint $x \in X_0$. The tangent space to the moduli stack $\mathcal{M}_B(X_\bullet, G)$ of G-local systems on $|X_\bullet|$ at L is $H^1(X_\bullet, \mathrm{ad}(L))$ where $\mathrm{ad}(L)$ is the adjoint local system, equal to $End(L)$ in the linear case and derived from the adjoint action of G on $Lie(G)$ in general. See [94] for a

discussion of some fine points on tangent spaces of moduli of local systems.

Combining differential forms on the various simplicial levels gives a complex of forms on a simplicial variety, as is known from [39] and [55]. This may be applied here. If $(E_\bullet, \theta)$ is a Higgs bundle on $X_\bullet$, define the Dolbeault cohomology $H^i_{\mathrm{Dol}}(X_\bullet, E_\bullet, \theta)$ to be the cohomology of the total complex obtained by adding together the Dolbeault complexes $A^\bullet_{\mathrm{Dol}}(X_k, E_k, \theta)$ (or any equivalent functorial family of complexes computing the same hypercohomology) on each X_k and adding the alternating sum of face maps to the differential. More generally if (E, ∇) is a bundle with λ-connection then we can define the de Rham cohomology $H^i_{\mathrm{DR}}(X_\bullet, E_\bullet, \nabla)$ using the de Rham complexes on each X_k. The simplicial version of Biswas and Ramanan's calculation of the deformation space [16] holds:

Lemma 7.11. *Suppose $(P_\bullet, \theta)$ is a principal G-Higgs bundle on $X_\bullet$ of semiharmonic type. Let $(\mathrm{ad}(P), \theta)$ denote the linear Higgs bundle obtained from the adjoint representation. The tangent space to the moduli stack $\mathcal{M}_H(X_\bullet, G)$ is naturally identified as $H^1_{\mathrm{Dol}}(X_\bullet, \mathrm{ad}(P), \theta)$. The corresponding statement holds for the relative tangent space to the moduli stack $\mathcal{M}_{\mathrm{Hod}}(X_\bullet, G)$ of λ-connections over any $\lambda \in \mathbb{A}^1$.*

Suppose $x \in X_0$ is a point. It may be viewed as a simplicial morphism $x \to X_\bullet$ from the constant one-point simplicial scheme to $X_\bullet$. The relative Dolbeault complex is the cone on the map

$$\bigoplus_{j,k} A^j_{\mathrm{Dol}}(X_k, E_k, \theta) \to E_0(x),$$

or equivalently the kernel of this map, giving a complex which calculates the cohomology relative to the basepoint. Again the same may be said for de Rham cohomology.

Remark 7.12. *The tangent space to $R_\eta(X_\bullet, x, G)$ is given by the relative cohomology $H^1_\eta(X_\bullet, x, \mathrm{ad}(\rho))$ of the required type.*

Proposition 7.13. *Suppose $X_\bullet$ is a simplicial smooth projective variety, connected, with basepoint $x \in X_0$. Suppose $V_\bullet$ is a polarizable variation of Hodge structure. Then the complete local ring $\widehat{\mathcal{O}}_{\rho,x}$ of the formal completion of the representation variety $R_B(X_\bullet, x, GL(n))$ at the monodromy representation ρ of $V_\bullet$ has a natural and functorial mixed Hodge structure generalizing that of* [42].

Proof. This is a sketch of proof. Suppose first that $\mathbf{x}_\bullet \to X_\bullet$ is a simplicial basepoint meeting all components of X_0, X_1 and X_2, so Proposition 6.3 applies. The complete local ring $\widehat{\mathcal{O}}_{\rho,\mathbf{x}_\bullet}$ of the formal completion of

$R_B(X_\bullet, \mathbf{x}_\bullet, GL(n))$ at ρ has a unique mixed Hodge structure compatible with the maps in the cartesian square (6.1) and the mixed Hodge structures on the local rings of the other three pieces given by [42]. This is because the maps in (6.1) are morphisms of mixed Hodge structures, and the local ring of the fiber product is the tensor product of the local rings of the other three pieces so it inherits an MHS.

Write $\mathbf{x}_\bullet = \langle x\rangle \sqcup \langle y^1\rangle \sqcup \cdots \sqcup \langle y^a\rangle$. Write $Y := \{y^1, \ldots, y^a\}$. At $y \in Y$ let V_y denote the Hodge structure fiber of $V_\bullet$ at y. This determines a mixed Hodge structure on the formal completion of $GL(V_y)$ at the identity. For these mixed Hodge structures, the action of $\prod_{y\in Y} GL(V_y)$ on $R_B(X_\bullet, \mathbf{x}_\bullet, GL(n))$ is compatible with the mixed Hodge structures. The action is free and

$$R_B(X_\bullet, x, GL(n)) = R_B(X_\bullet, \mathbf{x}_\bullet, GL(n))/\prod_{y\in Y} GL(V_y).$$

The complete local ring $\widehat{\mathcal{O}}_{\rho,x}$ is thus the subring of invariants in $\widehat{\mathcal{O}}_{\rho,\mathbf{x}_\bullet}$ under the formal action of $\prod_{y\in Y} GL(V_y)$, so there is an exact sequence

$$0 \to \widehat{\mathcal{O}}_{\rho,x} \to \widehat{\mathcal{O}}_{\rho,\mathbf{x}_\bullet} \to \widehat{\mathcal{O}}_{W,(\rho,1,\ldots,1)}$$

where

$$W = R_B(X_\bullet, \mathbf{x}_\bullet, GL(n)) \times \prod_{y\in Y} GL(V_y).$$

The map on the right is a map of MHS so $\widehat{\mathcal{O}}_{\rho,x}$ acquires a MHS. □

8. The normal case

A normal variety is geometrically unibranched. Conversely, if X is geometrically unibranched then its normalization $\tilde{X} \to X$ is a one-to-one map and induces an homeomorphism of topological realizations. This localizes in the etale topology so the same hold when X is a DM-stack.

A more general condition in the situation of a simplicial variety is the following "finite index condition".

Condition 8.1. *For $X_\bullet$ a simplicial smooth projective variety, the present condition says that:*
(1) for any two components X_0^i and X_0^j of X_0, there is a component X_1^{ij} of X_1 which dominates them by ∂_0 and ∂_1 respectively;
(2) for any basepoint $x \in X_0^i$, the image of $\pi_1(X_0^i, x) \to \pi_1(|X_\bullet|, x)$ has finite index; and
(3) every polarizable local system on $X_\bullet$ is strongly polarizable.

Condition (2) is independent of the choice of basepoint, because a path in X_0^i can be lifted to X_1^{ij} and projected to a path in X_0^j.

One could conjecture that Condition (3) is a consequence of the other two conditions, or perhaps some other natural geometric condition. I couldn't find an argument, but Theorem 8.4 will say that it holds for standard resolutions of geometrically unibranched proper DM-stacks.

Lemma 8.2. *If $X_\bullet$ is a simplicial smooth projective variety satisfying Condition 8.1, then the following conditions for a local system $L_\bullet$ on $X_\bullet$ are equivalent:*
(a) $L_\bullet$ is semisimple;
(b) there exists a component X_0' of X_0 such that $L_0|_{X_0'}$ is semisimple;
(c) $L_\bullet$ is polarizable;
(d) $L_\bullet$ is strongly polarizable.

For a G-torsor, the monodromy is reductive if and only if the monodromy of its restriction to X_0' is reductive.

Proof. Semisimplicity of a representation is equivalent to semisimplicity of its restriction to a finite-index subgroup. Therefore, from part (2) of 8.1, (b) implies (a) and (a) implies:
$(b)'$ for any component X_0' of X_0, $L_0|_{X_0'}$ is semisimple,
a condition which clearly implies (b). Also, (c) implies $(b)'$ since polarizability and semisimplicity are the same on a smooth quasiprojective variety. The pullback of a polarizable local system is again polarizable, so $(b)'$ implies (c). By part (3) of 8.1, (d) is equivalent to (c). □

Condition 8.1 holds in a wide variety of cases. In preparation for the proof, here is a version of Zariski's connectedness.

Lemma 8.3. *Suppose Z is a smooth variety with a projective map to a connected geometrically unibranched DM-stack X, such that every component of Z dominates X. Suppose $U \subset Z$ is a dense open subset. Then every connected component of $Z \times_X Z$ meets $U \times_X U$.*

Proof. Suppose $X' \to X$ is an etale covering, then the same statement for $Z \times_X X' \to X'$ implies the statement for $Z \to X$. Therefore we may assume that X is a quasiprojective scheme, also it can be assumed reduced.

Let $Z \xrightarrow{g} V \xrightarrow{h} X$ be the Stein factorization: h is finite and g has connected fibers. All irreducible components of V dominate X, so they have the same dimension as X. There is an open dense set $W \subset V$ such that U meets all the fibers $g^{-1}(w)$ for $w \in W$.

Suppose $(z_1, z_2) \in Z \times_X Z$, with $z_1, z_2 \mapsto x \in X$. Let $v_1 = g(z_1)$ and $v_2 = g(z_2)$. Thus $(v_1, v_2) \in V \times_X V$. Let N_1 and N_2 be small

usual analytic neighborhoods of v_1 and v_2 respectively in V. Their images $h(N_i) \subset X$ are germs of closed subvarieties of the same dimension as X, so by the hypothesis that X is geometrically unibranched, the $h(N_i)$ must contain neighborhoods of x. But $W \subset V$ is a dense Zariski-open subset, so $h(N_1 \cap W) \cap h(N_2 \cap W) \neq \emptyset$. By successively reducing the size of the neighborhoods, we can choose a sequence of points $(w_1(j), w_2(j))_{j \in \mathbb{N}} \in W \times_X W$ approaching (v_1, v_2) for $j \to \infty$. Lift $w_i(j)$ to points $y_i(j) \in U$. Since g is proper, a subsequence of $(y_1(j), y_2(j)) \in U \times_X U$ converges to some $(z_1', z_2') \in Z \times_X Z$ lying over $(v_1, v_2) \in V \times_X V$. But the fibers of g are connected (that is where Zariski's connectedness theorem is used), so z_1' is connected to z_1 in $g^{-1}(v_1)$ and z_2' is connected to z_2 in $g^{-1}(v_2)$. Therefore (z_1', z_2') lies in the same connected component of $Z \times_X Z$ as (z_1, z_2). However, (z_1', z_2') is also a limit of points in $U \times_X U$, so the component of (z_1, z_2) meets $U \times_X U$. □

Theorem 8.4. *Suppose that X is a proper singular DM-stack which is reduced, connected and geometrically unibranched, that is the analytic germ of an etale chart at any singular point is irreducible. Then, for a proper surjective hypercovering by smooth projective varieties $Z_\bullet \to Y$ constructed as in Theorem 5.8, the finite index condition 8.1 holds.*

Proof. We may assume that X is reduced (the reduced substack has the same topological type). Also it is irreducible, because connected and geometrically unibranched. If a hypercovering is constructed as in Theorem 5.8 then it satisfies condition (1). Suppose $V \to X$ is a surjective etale map. Elements of the fundamental group $\pi_1(X, x)$ can be viewed as paths which are piecewise continuous on V, related by the equivalence relation $V \times_X V$ at the jumping points. Furthermore the jumping points can be assumed general, i.e. where V is smooth. Since V is geometrically unibranched, paths can be moved away from the singularities and in fact, into any dense open substack. There exists an open dense substack $U \subset X$ which is smooth and a gerb over its smooth coarse moduli space $U^{\mathbf{c}}$. Then U is also connected and paths can be moved into U, so $\pi_1(U)$ surjects onto $\pi_1(X)$. Now suppose $f : Z_\bullet \to X$ is a proper surjective hypercovering, in particular by Proposition 5.15, $\pi_1(|Z_\bullet|) \cong \pi_1(X)$. Some connected component Z_0' dominates X, from which it follows that $\pi_1(f^{-1}(U) \cap Z_0') \to \pi_1(U^{\mathbf{c}})$ has image of finite index. Thus the image is of finite index in $\pi_1(U)$, and in turn the image of $\pi_1(Z_0')$ in $\pi_1(X) = \pi_1(|Z_\bullet|)$ has finite index.

For (3), given a polarizable local system $L_\bullet$ on $Z_\bullet$, we need to construct a strong polarization, that is a collection of harmonic metrics

h_k on L_k compatible with the restrictions. Eventually adding an extra component to $Z = Z_0$, we may assume that there is a component $Z^1 \subset Z$ containing an open set $U^1 \subset Z^1$ such that $U^1 \to X$ is a finite etale Galois cover over its image U_X which is in the smooth locus of X, and in fact in the locus where X is a gerb over the smooth part of $X^{\mathbf{c}}$.

Let Φ be the Galois group acting on U^1; by equivariant resolution of singularities [111] [9], we may assume that it extends to an action on Z^1. Therefore by averaging over Φ an initial choice of harmonic metric on L_{Z^1}, we obtain a Φ-invariant harmonic metric h' over U^1. For each component Z^i of Z, choose a component R^{1i} mapping by dominant maps $\partial_0 : R^{1i} \to Z^1$ and $\partial_1 : R^{1i} \to Z^i$. Then $\partial_1^*(L_{Z^i})$ is a local system on R^{1i} isomorphic by the descent datum, to $\partial_0^*(L_{Z^1})$. In general, given a harmonic metric on the pullback of a semisimple local system by a dominant map of smooth projective varieties, there is a unique harmonic metric downstairs whose pullback is the given one. So there is a harmonic metric h_i on L_{Z^i} with $\partial_1^*(h_i) = \partial_0^*(h_1)$ on R^{1i}. Together these define a harmonic metric h on L_0 over $Z = Z_0$.

The descent datum φ gives a continuous isomorphism between the two pullbacks $\mathrm{pr}_0^*(L_0)$ and $\mathrm{pr}_1^*(L_0)$ over $Z \times_X Z$ as pointed out in Remark 5.10. Let $K \subset Z \times_X Z$ be the subset of points (z_1, z_2) where $\varphi_* \mathrm{pr}_0^*(h) = \mathrm{pr}_1^*(h)$. The strong polarizability condition says that K should be all of $Z \times_X Z$.

The subset K is closed in the usual topology, since it results from the comparison of two continuously varying metrics. It is also open, indeed if $(z_1, z_2) \in K$ and (y_1, y_2) is a nearby point, then there is a connected smooth projective algebraic curve $C \to Z \times_X Z$ passing through (z_1, z_2) and (y_1, y_2). The two pullback metrics induce harmonic metrics on $L|_C$ which agree over (z_1, z_2), but a harmonic metric on a smooth projective variety is determined by its value at one point, so the two metrics agree over (y_1, y_2) too. This shows that $(y_1, y_2) \in K$, so K is open. It follows that K is a union of connected components of $Z \times_X Z$.

On the other hand, the invariance property of h_1 means essentially that over U_X it is pulled back from a harmonic metric on the local system L over U_X, so an argument with the descent data will show that the collection of h^i are compatible with the descent data on the open set $U \subset Z$ which is the inverse image of U_X. Thus $U \times_X U \subset K$. Lemma 8.3 implies that $K = Z \times_X Z$ so L is strongly polarizable. $\square$

If the condition "finite index" is replace by "surjective" then there is a closed immersion of representation spaces.

Lemma 8.5. *Suppose $X_\bullet$ is a simplicial smooth projective variety, connected, and let (X_0', x) be a connected component with basepoint in X_0.*

Suppose that $\pi_1(X'_0, x) \to \pi_1(X_\bullet, x)$ is surjective. Then the map

$$R_B(X_\bullet, x, G) \to R_B(X'_0, x, G)$$

is a closed immersion.

Proof. It is just given by the equations saying that the elements of the kernel of the map on fundamental groups, have trivial image. □

This lemma would apply, for example, to geometrically unibranched DM-stacks with quasiprojective moduli space and trivial generic stabilizer. At points corresponding to variations of Hodge structures, the closed immersion expresses the mixed Hodge structure on the complete local ring of $R_B(X_\bullet, x, G)$ as a quotient of that of $R_B(X'_0, x, G)$ by a mixed Hodge ideal.

Suppose $X_\bullet$ is a simplicial smooth projective variety, connected, satisfying the finite index condition 8.1. Suppose P is a G-principal bundle with λ-connection, or a G-torsor on $|X_\bullet|$. Fix a basepoint $x \in X_0$ and a framing for $P(x)$. One should be able to construct, following [55], a Lie algebra of forms with coefficients in $\mathrm{ad}(P)$ $A^\bullet_\eta(X_\bullet, \mathrm{ad}(P))$, augmented towards $Lie(G)$, which controls the deformation theory of P in the sense of Goldman-Millson. Say that a dgla is formal in degrees $\leq 1\frac{1}{2}$ if it is joined to a complex with zero differential, by morphisms inducing isomorphisms on H^0 and H^1 and injections on H^2. This is enough to get a control of the structure of the representation space.

Conjecture 8.6. *In this situation, if P is polarizable and $X_\bullet$ satisfies the finite index condition 8.1, then the above dgla is formal in degrees $\leq 1\frac{1}{2}$. Furthermore in this case there are natural quasiisomorphisms between the Dolbeault dgla controling deformations of the G-principal Higgs bundle and the de Rham and Betti dgla's controling deformations of the associated G-torsor.*

To get around this conjecture we can prove directly one of the main consequences, but without making any statement about quadraticity.

Lemma 8.7. *Suppose $X_\bullet$ is a simplicial scheme with smooth projective levels satisfying the finite index condition 8.1. Suppose $\rho : \pi_1(X_\bullet) \to G$ is a semisimple representation, corresponding to principal G-Higgs bundle (P, θ). The local analytic structures of $M_H(X_\bullet, G)$ at (P, θ) and of $M_B(X_\bullet, G)$ or $M_{DR}(X_\bullet, G)$ at ρ are the same.*

Proof. In the case of a smooth projective variety, the formal local structures of the representation spaces for $\eta = B$ and $\eta = H$, at points corresponding to a semisimple representation and its corresponding Higgs bundle, are canonically isomorphic. The isomorphism respects

the group actions by change of frames, and is functorial for morphisms of smooth projective varieties.

This comes from the formality isomorphism on the Goldman-Millson dgla's for a smooth projective variety. From the expression of (6.1) we get the same canonical isomorphisms

$$R_B(X_\bullet, \mathbf{x}_\bullet, GL(n))^{\wedge,\rho} \cong R_H(X_\bullet, \mathbf{x}_\bullet, GL(n))^{\wedge,(P,\theta)}.$$

From Condition 8.1, ρ is a point where the stabilizer is reductive. Using Luna's etale slice theorem as in [42], and taking the quotient by the stabilizer, gives the required local formal isomorphism $M_B^{\wedge,\rho} \cong M_H^{\wedge,(P,\theta)}$. □

Condition 8.1 implies that the categorical equivalence between Higgs bundles and local systems gives a homeomorphism of character varieties, joining together two different complex structures to give a quaternionic structure as in [48]. One expects that some condition such as 8.1 is necessary here, because of the non-continuity of the correspondence at non-semisimple points, see the Counterexample of [96] (II, p. 39).

Theorem 8.8. *Suppose $X_\bullet$ is a simplicial smooth projective variety, connected and which satisfies the finite index condition 8.1. Suppose G is a linear reductive group. Then the points of the various coarse moduli spaces $M_\eta(X_\bullet, G)$ parametrize polarizable G-local systems. The correspondence between Higgs bundles and local systems gives a homeomorphism of coarse moduli spaces*

$$M_H(X_\bullet, G)^{\mathrm{top}} \cong M_B(X_\bullet, G)^{\mathrm{top}}.$$

There are stratifications of M_H, M_{DR}, and M_B by locally closed smooth subvarieties which correspond to each other by the above homeomorphism and the Riemann-Hilbert isomorphism between M_{DR}^{an} and M_B^{an}, such that the Hitchin and Betti complex structures combine to give a hyperkähler structure on each stratum.

Proof. This is a sketch of proof. The correspondence preserves semisimplicity so it gives a map from the points of M_H to the points of M_B. Proceed as in [96] to get the homeomorphism, using the real subspaces of $R_B^h \subset R_B$ and $R_H^h \subset R_H$ consisting of framings compatible with harmonic metrics. The moduli spaces are quotients of R_B^h and R_H^h by compact groups. This argument will give, furthermore, that the map

$$M_\eta(X_\bullet, G) \to M_\eta(X_0, G)$$

is a proper map of topological spaces, from which it follows that it is a proper map of schemes. Since, for $\eta = B$, these are affine, we get in fact that the emap is finite.

Define a canonical stratification by starting with the open set of smooth points (of the reduced subscheme) where furthermore the restriction map to $M_\eta(X_0, G)$ is etale onto its image, looking at the complement, and continuing with the same construction. Lemma 8.7 shows that a point $\rho \in M_B$ will be at the same depth of this stratification as its corresponding point $(P, \theta) \in M_H$. The images of the strata are canonically defined locally closed subvarieties of $M_\eta(X_0, G)$. As such, they are compatible with all of the complex structures, so they are hyperkähler subvarieties of the hyperkähler structure of Hitchin-Fujiki [43]. Being etale over those of $M_\eta(X_0, G)$, the strata in $M_\eta(X_\bullet, G)$ have hyperkähler structures too. $\square$

The homeomorphism gives continuity of the $\mathbb{C}^*$ action.

Corollary 8.9. *Suppose $X_\bullet$ is a simplicial smooth projective variety, connected and which satisfies the finite index condition 8.1. Then the action of $\mathbb{C}^*$ is continuous on the character variety $M_B(X_\bullet, G)$.*

In particular, if ρ is a semisimple representation of $\pi_1(|X_\bullet|)$ which is locally rigid, then it is fixed by the action of $\mathbb{C}^*$ so it underlies a strongly polarizable variation of Hodge structure.

The real Zariski closure of its monodromy group is of Hodge type. Therefore, lattices in real groups not of Hodge type cannot occur as $\pi_1(|X_\bullet|)$.

Proof. The action is algebraic on M_H so by the homeomorphism of the previous theorem it is continuous on M_B. The rest follows as in [95]. For the last part, note that the real Zariski closure of the monodromy group of $\pi_1(X_0)$ has finite index in the real Zariski closure of the monodrom on $\pi_1(|X_\bullet|)$, so the conditions of Hodge type are equivalent, one concludes using [95] for X_0. $\square$

By Theorem 8.4, these restrictions, analogous to those for smooth projective varieties, apply in particular to any normal or even geometrically unibranched DM-stack.

An interesting question is whether other restrictions on fundamental groups of compact Kähler manifolds, including many works such as Gromov's [47]—see the full discussion of [3]—extend to the $\pi_1(|X_\bullet|)$ for $X_\bullet$ satisfying the finite index condition 8.1. A weaker question is to what extent these restrictions hold for smooth proper DM-stacks. Is the class of fundamental groups of smooth proper DM-stacks different from the classes of compact Kähler groups, or fundamental groups of smooth projective varieties? And how do these compare with the classes of fundamental groups of normal projective varieties, normal DM-stacks, the $\pi_1(|X_\bullet|)$ for $X_\bullet$ satisfying the finite index condition 8.1, etc?

9. The smooth case

Look now at the above constructions for the case when X is a smooth proper Deligne-Mumford stack. This was our main and original motivation, even though for expositional reasons we have concentrated on the simplicial case up until now. It is one of the cases which has attracted the most attention in the literature. For example, Biswas-Gómez-Hoffmann-Hogadi [12] treat local systems over an abelian gerb. If X is a smooth projective variety with simple normal crossings divisor D, then the Cadman-Vistoli root stacks which have been discussed previously are smooth and proper. Local systems on root stacks correspond to parabolic bundles (with rational weights), so the numerous works concerning parabolic bundles may be viewed as treating local systems on the root stacks, as will be discussed in detail in the second half of this section.

Fix a connected smooth proper DM-stack X, and let $Z_\bullet \to X$ be a proper surjective hypercovering such that the Z_k are smooth projective varieties given by Theorem 5.8. The first terms (Z, R, K) are assumed to form a partial simplicial resolution constructed according to the recipe above Theorem 5.8, starting from a surjective-where-etale morphism $Z \to X$ from a smooth projective variety of Theorem 5.4.

For $\eta = B, DR, H, Hod, DH$ the moduli stacks $\mathcal{M}_\eta(Z_\bullet, G)$ may be interpreted as moduli stacks of the various kinds of local systems on X

$$\mathcal{M}_\eta(Z_\bullet, G) \cong \mathcal{M}_\eta(X, G),$$

indeed bundles with λ-connection (resp. local systems) on $Z_\bullet$ descend to bundles with λ-connection (resp. local systems) on X, by Lemma 5.11 (resp. Lemma 5.14). Semistability for Higgs bundles requires some further discussion below.

Letting $z \in Z$ be a lift of the basepoint $x \in X$, the same may be said of the representation schemes

$$R_\eta(Z_\bullet, z, G) \cong R_\eta(X_\bullet, x, G).$$

Local systems on X may be identified with representations of Noohi's fundamental group $\pi_1(X, x)$ defined in [78], which is the same as the fundamental group of the topological realization $|X|$. So the Betti moduli stacks can be expressed

$$R_B(X, x, G) = Hom(\pi_1(X, x), G)$$

$$\mathcal{M}_{\mathrm{B}}(X) = Hom(\pi_1(X, x), G)/\!/G.$$

We have the Riemann-Hilbert correspondence between local systems and vector bundles with integrable algebraic connection

$$\mathcal{M}_{\mathrm{B}}(X)^{\mathrm{an}} \cong \mathcal{M}_{\mathrm{DR}}(X)^{\mathrm{an}}$$

which may be constructed over Z and then descended down to X.

A smooth proper DM-stack satisfies Condition 8.1, by Theorem 8.4, so polarizability, strong polarizability and semistability are the same by Lemma 8.2. More generally all the results of the previous section apply.

In order to give an intrinsic description of the moduli stack of Higgs bundles $\mathcal{M}_H(X,G)$, a notion of semistability is needed.

Nironi has introduced a very interesting notion of projective DM-stack [75]. This allows him to generalize the theory of moduli of vector bundles and similar objects, by applying the same techniques as in the projective case. Our technique applies to any proper smooth DM-stack, but doesn't give as much as what Nironi can do: we are constrained to consider only moduli spaces of objects with vanishing Chern classes, which correspond in some way to representations of the fundamental group, while Nironi's techniques in the case of a "projective" DM-stack (in his sense) would allow consideration of moduli spaces of vector bundles with arbitrary Chern classes.

On a general smooth proper DM-stack X we don't have a Kähler class to use for defining semistability, but due to the fact that we are interested in flat bundles here i.e. $c_2 = 0$, there are various ways of getting around that: either require semistability for *some* variety mapping to X, or for *all* varieties mapping to X.

Definition 9.1. *Suppose (E,θ) is a Higgs bundle on a smooth proper DM-stack X. We say that it is potentially semistable (resp. potentially polystable) if there exists a polarized projective variety Y and a surjective map $g : Y \to X$ such that the Higgs bundle $g^*(E,\theta)$ is slope-semistable (resp. slope-polystable) on Y with respect to the given polarization.*

In general this notion will not be very well behaved: even if X is a projective variety itself, we are allowing semistability with respect to an arbitrary polarization. However, when the Chern classes vanish then the condition no longer depends on a choice of polarization so we can expect that it gives a reasonable condition on a DM-stack too. Recall that Vistoli's theorem provides the notion of rational Chern classes on X, see [53]. Thus, the condition $c_i(E) = 0$ in $H^{2i}(|X|,\mathbb{Q})$ makes good sense.

The following condition for Higgs bundles has been introduced and extensively considered by Bruzzo, Hernández, Otero and others [23] [24]. They relate it to a condition of numerical effectivity, as was originally considered for vector bundles by Demailly, Peternell, and Schneider [35].

Definition 9.2. *Suppose (E, θ) is a Higgs bundle on a smooth proper DM-stack X. We say that it is pluri-semistable (resp. pluri-polystable) if for every curve Y and map $g : Y \to X$ the Higgs bundle $g^*(E, \theta)$ is slope-semistable (resp. slope-polystable) on Y with respect to the polarization which, for a curve, is unique up to scalars.*

Remark 9.3. *If (E, θ) is pluri-semistable (resp. pluri-polystable) then for any polarized smooth projective variety Y and map $g : Y \to X$, $g^*(E, \theta)$ is slope-semistable (resp. slope-polystable) on Y with respect to the given polarization. In particular (E, θ) is potentially semistable (resp. potentially polystable).*

Potential semistability implies pluri-semistability when the rational Chern classes vanish, and these conditions are also related to Higgs-nefness of the bundle and its dual, see Bruzzo-Otero [24, Theorem 4.7].

Lemma 9.4. *Suppose (E, θ) is a potentially semistable (resp. potentially polystable) Higgs bundle on X, with $c_i(E) = 0$ in rational cohomology for $i = 1, 2$. Then it is pluri-semistable (resp. pluri-polystable). In particular for any map from a smooth projective variety $g : Y \to X$, the pullback $g^*(E, \theta)$ is a successive extension of stable Higgs bundles and corresponds to a representation of $\pi_1(Y)$ via* [95]. *The rational Chern classes vanish for all i.*

Bruzzo and co-authors have formulated the following partial converse (for instance it would be the implication in the other direction in Bruzzo-Otero [24, Theorem 4.7]), which we call the *Bruzzo conjecture*:

Conjecture 9.5 (Bruzzo conjecture)**.** *If (E, θ) is a pluri-semistable Higgs bundle over a smooth proper DM-stack, then $c_i(E) = 0$ in rational cohomology for all i.*

This conjecture would generalize to Higgs bundles the theorem of Demailly, Peternell and Schneider who prove it for vector bundles i.e. when $\theta = 0$ [35].

After this discussion of semistability, we can formulate more precisely the moduli problems solved by $\mathcal{M}_{\mathrm{H}}(X, G)$ and $\mathcal{M}_{\mathrm{Hod}}(X, G)$.

Definition 9.6. *A G-principal Higgs bundle on X is of semiharmonic (resp. harmonic) type, if its Chern classes vanish in rational cohomology, and if it is potentially or equivalently pluri-semistable (resp. pluri-polystable). This definition extends to λ-connections too.*

If P is a G-principal Higgs bundle on X then its pullback to $Z_\bullet$ is of semiharmonic type if and only if P is. Hence, the moduli stack $\mathcal{M}_{\mathrm{H}}(X,G)$ parametrizes principal Higgs G-bundles of semiharmonic type; and the moduli stack $\mathcal{M}_{\mathrm{Hod}}(X,G) \to \mathbb{A}^1$ parametrizes principal G-bundles with λ-connection of semiharmonic type.

Theorem 9.7. *Proposition 7.2 gives a tannakian Kobayashi-Hitchin correspondence between Higgs bundles of semiharmonic type on X and local systems on X. The Higgs bundles of harmonic type correspond to the semisimple local systems, these conditions being the same as (strong) polarizability on both sides. For these polarizable objects, harmonic metrics exist which set up the correspondence via the same differential-geometric structures as in the case of varieties, over the etale local charts. The resulting map between moduli spaces is a homeomorphism and determines a hyperkähler structure.*

Proof. By Condition 8.1 and Lemma 8.2, polarizability, strong polarizability and semisimplicity are equivalent in the categories of local systems or Higgs bundles of semiharmonic type. Those tannakian categories are equivalent by Proposition 7.2. Given a Higgs bundle of harmonic type, its pullback to each Z_k is of harmonic type, so it has a unique structure of harmonic bundle. Furthermore, by strong polarizability, a compatible collection of metrics h_k may be chosen. Then from the condition that $Z \to X$ is surjective where etale, and the subsequent choice of the rest of $Z_\bullet$, the bundle, the harmonic metric, and various connection operators descend to X. Over etale charts in X, in particular those which are contained in Z, these structures satisfy the usual axioms for a harmonic metric. They give in particular the corresponding flat connection. The same discussion works starting from a semisimple local system. For the homeomorphism and hyperkähler structure, apply Theorem 8.8. □

Suppose X is a smooth variety and $D \subset X$ is a divisor with normal crossings. Hermitian Yang-Mills theory and the Kobayashi-Hitchin correspondence have been considered for parabolic bundles on (X,D) by many authors [15] [64] [74] [70] [71] [84] [100]. These theories may be related to the the corresponding theories over a smooth proper Cadman-Vistoli root stack, something that was basically observed by Daskalopoulos and Wentworth quite some time ago [29].

Let $Z \to X$ be the root stack corresponding to denominators n_i for the irreducible components D_i of D. As in the original article of Seshadri [93], a vector bundle on Z corresponds to a parabolic bundle on (X, D) such that the weights along D_i are in $\frac{1}{n_i}\mathbb{Z}$. This correspondence has been used and studied by many authors, see for example Boden [18], Balaji *et al* [5], Biswas [11], Borne [20] [21] as well as [53] and [54].

An important condition for a parabolic structure is to be locally abelian, that is near any multiple intersection point of D, the parabolic structure should decompose as a direct sum of parabolic line bundles. Borne and Vistoli [20] [21] have recently improved our understanding of this condition by the following result.

Theorem 9.8 (Borne). *Suppose $E = \{E_{\alpha_1,\ldots,\alpha_m}\}$ is a parabolic torsion-free sheaf (that is a system of torsion-free sheaves and inclusions satisfying the conditions of semicontinuity and twisting by the divisor components). Then E is a locally abelian parabolic bundle, if and only if all of the component sheaves $E_{\alpha_1,\ldots,\alpha_m}$ are locally free.*

Proof. If E is locally abelian then automatically the components are bundles, so the task is to prove that if each $E_{\alpha_1,\ldots,\alpha_m}$ is a vector bundle, then the parabolic structure is locally abelian.

This is Borne's Proposition 2.3.10 [21]. For the proof, he uses the following main statement which he attributes to Vistoli [21, Lemma 2.3.11]: suppose $E \subset F \subset E(D)$ is a pair of inclusions of locally free sheaves, with D a smooth divisor. Then F/E and $E(D)/F$ are locally free sheaves on D. The proof in turn refers to the formula of Auslander-Buchsbaum in EGA. □

A *parabolic λ-connection* is a locally abelian parabolic bundle $E_\bullet$ together with a λ-connection operator

$$\nabla : E_{\alpha_1,\ldots,\alpha_m} \to E_{\alpha_1,\ldots,\alpha_m} \otimes \Omega^1_X(\log D).$$

One defines the parabolic degree and hence the notion of parabolic stability. Moduli spaces for parabolic vector bundles, parabolic Higgs bundles, and parabolic connections have been studied in many places: [93] [69] [67] [19] [114] [76] [60] [74] [102] [4] [51] is a certainly non-exhaustive list.

Given a semistable parabolic λ-connection, the residual data are locally constant along the non-intersection points y of the divisor components D_i. Thus one can speak of the residue of (E, ∇) along D_i. It

is a pair

$$\mathrm{res}_{D_i,y}(E,\nabla) = \left(\bigoplus_{\alpha\in(-1,0]} \mathrm{gr}^{D_i}_{\alpha}(E(y)), \mathrm{res}(\nabla) \right)$$

consisting of a vector space graded by a finite number of parabolic weights $\alpha \in (-1,0]$, together with an endomorphism $\mathrm{res}(\nabla)$. The graded piece $\mathrm{gr}_{\alpha}(E(y))$ is the fiber at y of the quotient $E_{\alpha}/E_{\alpha-\epsilon}$, and the residue of ∇ comes from the action on this graded piece. Here y is in a single divisor component D_i so the parabolic structure near y is reduced to a single index, indicated for the notation by a superscript gr^{D_i}.

Say that (E,∇) has semisimple residues, if the $\mathrm{res}(\nabla)$ are semisimple endomorphisms. Note that this is a weaker condition than asking that the residue be semisimple for ∇ considered as a logarithmic connection on one of the component vector bundles $E_{\alpha_1,\ldots,\alpha_m}$, because this bigger residual endomorphism might have a unipotent factor which acts by strictly decreasing the parabolic weight.

One can more generally define the notion of parabolic bundle on a smooth DM-stack with respect to a normal crossings divisor, a viewpoint which is useful for the inductive kind of argument used in [54]. On the other hand, a parabolic bundle all of whose weights are integers, may be viewed as a usual parabolic bundle.

The bundles with λ-connection on the root stack $Z = X[\frac{D_1}{n_1},\ldots,\frac{D_m}{n_m}]$ are exactly the pullbacks of parabolic bundles from (X,D) such that the pullback has integer weights and trivial residue of the connection. Making this condition explicit gives the following proposition.

Proposition 9.9. *Suppose $\lambda\in\mathbb{C}$ and n_i are strictly positive integers. Pullback gives an equivalence of categories, preserving the conditions of (semi)stability and the Chern classes, between:*
—parabolic λ-connections on (X,D) such that the parabolic weights along D_i are in $\frac{1}{n_i}\mathbb{Z}$ and the residue of the connection on each parabolic graded piece is semisimple with a single eigenvalue given as follows:

$$\mathrm{res}^{D_i}_{\alpha}(\nabla) = \lambda\alpha\cdot 1_{\mathrm{gr}^{D_i}_{\alpha}(E)};$$

and
—bundles with λ-connection on the root stack $Z = X\left[\frac{D_1}{n_1},\ldots,\frac{D_m}{n_m}\right]$.

One may translate using this equivalence between the parabolic and stack-theoretic points of view, in particular the numerous works on harmonic theory and moduli for parabolic bundles become relevant for the problem we are considering here. Particularly so in the basic case

of a root stack. A further discussion of the details, such as the behavior of the harmonic metrics near D_i, would take us too far afield and these aspects are amply treated already in the many available references.

The analogue of parabolic structures for principal G-bundles is not completely straightforward: one needs to introduce the notion of parahoric structure, and this is the subject of current ongoing research by several authors [17] [6].

For smooth proper X it is natural to formulate Poincaré duality. The importance of Poincaré duality for the study of fundamental groups has become apparent in recent works of Bruno Klingler. The coarse moduli space of a smooth proper DM-stack X is a proper rational homology manifold. The cohomology of the stack is the same as that of its coarse moduli space, so it is easy to see that Poincaré duality holds for $H^{\bullet}(X, \mathbb{Q})$. This has been remarked for example by Abramovich, Graber and Vistoli in [1], and was undoubtedly one of the reasons for Deligne's comment about rational homology manifolds in [33]. Still, for cohomology with coefficients in a local system it is better to have an intrinsic proof such as was given by Behrend.

Theorem 9.10. *Suppose X is a connected smooth proper DM-stack of dimension n. Then the fundamental class of X gives a canonical isomorphism $H^{2n}(X, \mathbb{C}) \cong \mathbb{C}$; and for any local system L on X, the cup product followed by the trace $L \otimes L^* \to \mathbb{C}$ gives a perfect pairing*

$$H^i(X, L) \times H^{2n-i}(X, L^*) \to H^{2n}(X, \mathbb{C}) \cong \mathbb{C}.$$

Proof. We refer to Behrend [7]. □

Poincaré duality allows us to prove the purity of the mixed twistor structure on cohomology.

Corollary 9.11. *Suppose X is a connected smooth proper DM-stack. If $f : Z \to X$ is a dominant morphism from another smooth proper DM-stack (in particular Z could be a smooth projective variety) then for any local system L, pullback along f is an injection*

$$f^* : H^i(X, L) \hookrightarrow H^i(Z, f^*L).$$

If L is a pure variation of Hodge structure of weight w, then the mixed Hodge structure on $H^i(X, L)$ is pure of weight $i + w$.

Proof. Suppose $\dim(X) = n$ and $p : Y \to Z$ is a surjective morphism from a connected smooth projective variety, also of dimension n. This exists by Theorem 5.4. There is an open subset $U \subset X$ over which p is a finite etale covering of degree d. A top degree cohomology class on

X may be represented by a form which is compactly supported in U, so the pullback map

$$\mathbb{C} \cong H^{2n}(X, \mathbb{C}) \xrightarrow{p^*} H^{2n}(Y, \mathbb{C}) \cong \mathbb{C}$$

is multiplication by d. If $p_!$ denotes the Poincaré dual of p^* the standard argument shows that $p_!(p^*u) = d \cdot u$ for $u \in H^i(X, L)$, implying that p^* is injective.

Suppose $f : Z \to X$ is a dominant morphism of smooth proper DM-stacks. Choose a surjective map $q : V \to Z$ from a smooth projective variety with $\dim(V) = \dim(Z)$. Let Y be a general complete intersection of hyperplane sections in Z, with $\dim(Y) = \dim(X)$. The projection $p : Y \to X$ is surjective so by the previous discussion p^* is injective; it follows that f^* is injective.

If L is a variation of pure Hodge structure of weight w, the pullback map

$$p^* : H^i(X, L) \to H^i(Y, p^*(L))$$

is an injective morphism of mixed Hodge structures, whose target is pure of weight $i + w$, therefore $H^i(X, L)$ is pure of weight $i + w$. $\square$

10. Mixed twistor theory

Deligne's theory of [33] goes over to mixed twistor structures. This is useful for looking at the topology of simplicial smooth projective varieties, so we gives some details here expanding upon the places where it was mentioned in [99]. It will allow us to generalize Corollary 9.11 to a purity statement, Corollary 10.9, for any semisimple local system. The development presented here is undoubtedly subsumed in a theory of "mixed twistor modules" generalizing Saito's mixed Hodge modules as was done by Sabbah for the pure case [87].

A D-mixed twistor complex is a filtered complex of sheaves of $\mathcal{O}_{\mathbb{P}^1}$-modules $(\mathcal{F}^\bullet, W_\bullet)$ on $\mathbb{P}^1$ such that

$$H^i(W_n\mathcal{F}^\bullet / W_{n-1}\mathcal{F}^\bullet)$$

is a semistable vector bundle of slope $n + i$ on $\mathbb{P}^1$, nonzero for only finitely many (i, n).

A B-mixed twistor complex is a filtered complex of sheaves of $\mathcal{O}_{\mathbb{P}^1}$-modules $(\mathcal{F}^\bullet, W_\bullet)$ on $\mathbb{P}^1$ such that

$$H^i(W_m\mathcal{F}^\bullet / W_{m-1}\mathcal{F}^\bullet)$$

is a semistable vector bundle of slope m on $\mathbb{P}^1$, nonzero for only finitely many (i, m).

In our notations D stands for Deligne and B for Beilinson: the D-mixed Hodge complexes were defined by Deligne [33], whereas Beilinson's treatment [8], see also Huber [50], refers to the B-mixed notion. See also Zucker [115], where the notion of relaxed MHC is introduced.

I have often wondered about how to express the relationship between these two notions. Although this materiel is well-known to experts, it seems likely that some readers will find it useful to review the relationship. This explanation is easier to follow in the case of mixed twistor structures, since we can work within the abelian category of sheaves on $\mathbb{P}^1$, avoiding concerns about strictness of maps between filtered vector spaces.

Consider first the passage from a D-mixed twistor complex to the mixed twistor structure on cohomology. Recall that the spectral sequence of a filtered complex $(\mathcal{F}^\bullet, W_\bullet)$ has

$$E_0^{k,l} := W_{-k}\mathcal{F}^{k+l}/W_{-k-1}\mathcal{F}^{k+l}$$

with differential $d_0 : E_0^{k,l} \to E_0^{k,l+1}$ induced by the differential d of $\mathcal{F}^\bullet$. Then

$$E_1^{k,l}(\mathcal{F}^\bullet, W_\bullet) = H^{k+l}(W_{-k}/W_{-k-1}).$$

The differential $d_1 : E_1^{k,l} \to E_1^{k+1,l}$ is, with different indices, the connecting map

$$H^i(W_m/W_{m-1}) \to H^{i+1}(W_{m-1}/W_{m-2})$$

coming from the short exact sequence of complexes

$$0 \to W_{m-1}/W_{m-2} \to W_m/W_{m-2} \to W_m/W_{m-1} \to 0.$$

Going back to the indices k, l we obtain the expression

(10.1)

$$E_2^{k,l}(\mathcal{F}^\bullet, W_\bullet) = \frac{\ker\left(H^{k+l}(W_{-k}/W_{-k-1}) \to H^{k+l+1}(W_{-k-1}/W_{-k-2})\right)}{\mathrm{im}\left(H^{k+l-1}(W_{1-k}/W_{-k}) \to H^{k+l}(W_{-k}/W_{-k-1})\right)}.$$

The next differential is

$$d_2 : E_2^{k,l} \to E_2^{k+2,l-1}.$$

Finally, the spectral sequence abuts to $H^{k+l}(\mathcal{F}^\bullet)$ with the filtration induced by $W_\bullet$, more precisely defined as

$$W_m H^i(\mathcal{F}^\bullet) := \mathrm{im}\left(H^i(W_m\mathcal{F}^\bullet) \to H^i(\mathcal{F}^\bullet)\right).$$

This all works in the context of filtered complexes in any abelian category. In our case we work with the abelian category of sheaves of $\mathcal{O}_{\mathbb{P}^1}$-modules over $\mathbb{P}^1$.

If $(\mathcal{F}^\bullet, W_\bullet)$ is a D-mixed twistor complex, then by definition

$$E_1^{k,l}(\mathcal{F}^\bullet, W_\bullet) = H^{k+l}(W_{-k}/W_{-k-1})$$

is a semistable vector bundle of slope $(k+l-k)=l$ on $\mathbb{P}^1$. In particular the differential $d_1 : E_1^{k,l} \to E_1^{k+1,l}$ is a strict morphism between semistable vector bundles of the same slope l, so its kernel and cokernels are also semistable vector bundles of slope l, and indeed the expression (10.1) expresses $E_2^{k,l}$ as the quotient of a semistable bundle by another one of the same slope.

Corollary 10.1. *If $(\mathcal{F}^\bullet, W_\bullet)$ is a D-mixed twistor complex, then d_1 is a strict morphism between semistable vector bundles of slope l, and the cohomology $E_2^{k,l}$ of the resulting complex is a semistable vector bundle of slope l. Furthermore, $d_r = 0$ for all $r \geq 2$ and the spectral sequence degenerates at E_2. We obtain the expression*

$$(10.1) = E_2^{k,l} = \frac{W_{-k}H^{k+l}(\mathcal{F}^\bullet)}{W_{-k-1}H^{k+l}(\mathcal{F}^\bullet)}.$$

Hence, if we set

$$W_m^B H^i(\mathcal{F}^\bullet) := W_{m-i}H^i(\mathcal{F}^\bullet) = \mathrm{im}\left(H^i(W_m\mathcal{F}^\bullet) \to H^i(\mathcal{F}^\bullet)\right)$$

then $(H^i(\mathcal{F}^\bullet), W_\bullet^B)$ is a mixed twistor structure.

Proof. The first sentence is what was seen above. But then d_2 is a map from a semistable vector bundle of slope l to a semistable vector bundle of slope $l-1$, so $d_2 = 0$. Hence $E_3 = E_2$, d_3 becomes a morphism from a bundle of slope l to one of slope $l-2$ so it vanishes, and so on. Inductively $d_r = 0$ for all $r \geq 2$ so the spectral sequence degenerates at E_2. Thus $E_2^{k,l} = Gr_{-k}^W H^{k+l}$ and this is a semistable bundle of slope l. Changing the indices, this says that $Gr_m^W H^i$ is semistable of slope $m+i$. To get a mixed twistor structure we have to shift the filtration; the new filtration may be denoted $W_\bullet^B$ because it will coincide with the filtration obtained from the Beilinson picture (see below). We have

$$Gr_m^{W^B} H^i = Gr_{m-i}^W H^i$$

which is semistable of slope $(m-i+i)=m$, which exactly says that $H^i(\mathcal{F}^\bullet)$ together with its filtration $W_\bullet^B$ is a mixed twistor structure. □

We now look at how this works if we first pass to the Beilinson point of view.

Corollary 10.2. *If $(\mathcal{F}^\bullet, W_\bullet^B)$ is a B-mixed twistor complex then the spectral sequence degenerates at*

$$E_1^{k,l}(\mathcal{F}^\bullet, W_\bullet^B) = H^{k+l}(Gr_{-k}^{W^B}\mathcal{F}^\bullet)$$

which are semistable bundles of slope $-k$ on $\mathbb{P}^1$. The induced filtration

$$W_m^B H^i(\mathcal{F}^\bullet) := \mathrm{im}\left(H^i(W_m^B\mathcal{F}^\bullet) \to H^i(\mathcal{F}^\bullet)\right)$$

gives a mixed twistor structure on $H^i(\mathcal{F}^\bullet)$*.*

Proof. Follow the proof of Corollary 10.1, noting that $H^{k+l}(Gr^{W^B}_{-k}\mathcal{F}^\bullet)$ is semistable of slope $-k$ by the definition of B-mixed twistor complex. The differential d_1 vanishes already because it decreases the slope. □

Given a D-mixed twistor complex $(\mathcal{F}^\bullet, W_\bullet)$ with the differential of the complex $\mathcal{F}^\bullet$ denoted by d, we obtain a B-mixed twistor complex $(\mathcal{F}^\bullet, W^B_\bullet)$ by setting

$$W^B_m(\mathcal{F}^i) :=$$
$$\ker\left(d : W_{m-i}\mathcal{F}^i \to \frac{W_{m-i}\mathcal{F}^{i+1}}{W_{m-i-1}\mathcal{F}^{i+1}}\right).$$

This is a subobject of $W_{m-i}\mathcal{F}^i$. Note in passing that if $d = 0$ this is just the same as W_{m-i} explaining the notation in Corollary 10.1 above. The filtration $W^B_\bullet$ is usually called $Dec(W_\bullet)$, cf [33] and [50].

Lemma 10.3. *The above construction starting from a D-mixed twistor complex yields a B-mixed twistor complex. More particularly,*

$$E_1^{k,l}(\mathcal{F}^\bullet, W^B_\bullet) = H^{k+l}(Gr^{W^B}_{-k}\mathcal{F}^\bullet) = E_2^{2k+l,-k}(\mathcal{F}^\bullet, W_\bullet).$$

The spectral sequence for $(\mathcal{F}^\bullet, W^B_\bullet)$ *degenerating at* E_1 *abuts to* W^B *of Corollary 10.2 which in this case is the same filtration as the* W^B *of Corollary 10.1. Furthermore the construction* $W \mapsto W^B$ *is multiplicative: if* $(\mathcal{F}^\bullet, W_\bullet)$ *and* $(\mathcal{G}^\bullet, W_\bullet)$ *are D-mixed twistor complexes then*

$$W^B_\bullet(\mathcal{F}^\bullet \otimes \mathcal{G}^\bullet)$$

is the tensor product filtration of $W^B_\bullet$ *on* $\mathcal{F}^\bullet$ *and* $W^B_\bullet$ *on* $\mathcal{G}^\bullet$*.*

Proof. Unfortunately the E_0 term for W^B doesn't coincide with the E_1 term for W, instead there is an extra acyclic complex there. This is just Proposition 1.3.4 of [32] applied to the abelian category of sheaves of $\mathcal{O}_{\mathbb{P}^1}$-modules, but we write things out more explicitly. See also the discussion of Lemma 1.3.15 of [32], as well as certainly other more recent references.

We have

$$E_0^{k,l}(\mathcal{F}^\bullet, W^B_\bullet) = \frac{W^B_{-k}\mathcal{F}^{k+l}}{W^B_{-k-1}\mathcal{F}^{k+l}}$$
$$= \frac{\ker\left(d : W_{-2k-l}\mathcal{F}^{k+l} \to W_{-2k-l}\mathcal{F}^{k+l+1}/W_{-2k-l-1}\right)}{\ker\left(d : W_{-2k-l-1}\mathcal{F}^{k+l} \to W_{-2k-l-1}\mathcal{F}^{k+l+1}/W_{-2k-l-2}\right)}.$$

In particular there is a natural projection

$$E_0^{k,l}(\mathcal{F}^\bullet, W^B_\bullet) \to H^{k+l}(Gr^W_{-2k-l}\mathcal{F}^\bullet) = E_1^{2k+l,-k}(\mathcal{F}^\bullet, W_\bullet)$$
$$= \frac{\ker\left(d : W_{-2k-l}\mathcal{F}^{k+l}/W_{-2k-l-1} \to W_{-2k-l}\mathcal{F}^{k+l+1}/W_{-2k-l-1}\right)}{\mathrm{im}\left(d : W_{-2k-l}\mathcal{F}^{k+l-1}/W_{-2k-l-1} \to W_{-2k-l}\mathcal{F}^{k+l}/W_{-2k-l-1}\right)}.$$

Let $\mathcal{U}^{k,l}$ be the kernel, that is we have an exact sequence

$$0 \to \mathcal{U}^{k,l} \to E_0^{k,l}(\mathcal{F}^\bullet, W^B_\bullet) \to E_1^{2k+l,-k}(\mathcal{F}^\bullet, W_\bullet) \to 0.$$

In the above expressions, look at the subobject of $W^B_{-k}\mathcal{F}^{k+l}/W^B_{-k-1}$ determined by the image of $W_{-2k-l-1}$. It is contained in $\mathcal{U}^{k,l}$, and is of the form $W_{-2k-l-1}\mathcal{F}^{k+l}/\ker(d)$ which is naturally isomorphic to the image, denoted by $\mathrm{im}(Gr^W_{-2k-l-1}(d^{k+l})$, of

$$d : Gr^W_{-2k-l-1}\mathcal{F}^{k+l} \to Gr^W_{-2k-l-1}\mathcal{F}^{k+l+1}.$$

On the other hand,

$$\frac{W^B_{-k}\mathcal{F}^{k+l}}{W_{-2k-l-1}\mathcal{F}^{k+l}} = \ker\left(d : Gr^W_{-2k-l}\mathcal{F}^{k+l} \to Gr^W_{-2k-l}\mathcal{F}^{k+l+1}\right).$$

The kernel of the projection from here to $H^{k+l}(Gr^W_{-2k-l}\mathcal{F}^\bullet)$ is by definition the image denoted $\mathrm{im}(Gr^W_{-2k-l}(d^{k+l-1})$ of

$$d : Gr^W_{-2k-l}\mathcal{F}^{k+l-1} \to Gr^W_{-2k-l}\mathcal{F}^{k+l}$$

This leads to an exact sequence

$$0 \to \mathrm{im}(Gr^W_{-2k-l-1}(d^{k+l})) \to \mathcal{U}^{k,l} \to \mathrm{im}(Gr^W_{-2k-l}(d^{k+l-1})) \to 0.$$

On the other hand, note that

$$d : W_{-2k-l-1}\mathcal{F}^{k+l-1} \to \{0\} \subset E_0^{k,l}(\mathcal{F}^\bullet, W^B_\bullet)$$

so d induces a map

$$Gr^W_{-2k-l-1}\mathcal{F}^{k+l-1} \to \mathcal{U}^{k,l}.$$

The kernel of $d : Gr^W_{-2k-l-1}\mathcal{F}^{k+l-1} \to Gr^W_{-2k-l-1}\mathcal{F}^{k+l}$ maps to zero in $\mathcal{U}^{k,l}$ in view of the original expression for $E_0^{k,l}(\mathcal{F}^\bullet, W^B_\bullet)$. Thus d induces a map

$$\mathrm{im}(Gr^W_{-2k-l-1}(d^{k+l-1}) \to \mathcal{U}^{k,l}.$$

This splits the previous exact sequence, so we get a direct sum decomposition

$$\mathcal{U}^{k,l} = \mathrm{im}(Gr^W_{-2k-l-1}(d^{k+l})) \oplus \mathrm{im}(Gr^W_{-2k-l}(d^{k+l-1})).$$

The differential

$$d_0 : E_0^{k,l}(\mathcal{F}^\bullet, W^B_\bullet) \to E_0^{k,l+1}(\mathcal{F}^\bullet, W^B_\bullet)$$

sends $\mathcal{U}^{k,l}$ to $\mathcal{U}^{k,l+1}$ and on there it is equal to the splitting map defined above, identifying

$$\mathrm{im}(Gr^W_{-2k-l}(d^{k+l-1}) \subset \mathcal{U}^{k,l}$$

with

$$\mathrm{im}(Gr^W_{-2k-l}(d^{k+l-1}) \subset \mathcal{U}^{k,l+1}.$$

It follows that the complex

$$\cdots \stackrel{d_0}{\to} \mathcal{U}^{k,l-1} \stackrel{d_0}{\to} \mathcal{U}^{k,l} \stackrel{d_0}{\to} \cdots$$

is acyclic. The map

$$E_0^{k,l}(\mathcal{F}^\bullet, W^B_\bullet) \to E_1^{2k+l,-k}(\mathcal{F}^\bullet, W_\bullet)$$

which is compatible with the differential d_0 on the left and d_1 on the right, so for k fixed it induces a map of complexes. The kernel of this map of complexes is the acyclic complex formed by the $\mathcal{U}^{k,l}$. We get an isomorphism on cohomology, which is to say an isomorphism between the next terms in the spectral sequence:

$$E_1^{k,l}(\mathcal{F}^\bullet, W^B_\bullet) \stackrel{\cong}{\to} E_2^{2k+l,-k}(\mathcal{F}^\bullet, W_\bullet).$$

Using our hypothesis that $(\mathcal{F}^\bullet, W_\bullet)$ is a D-mixed twistor complex, recall from Corollary 10.1 that $E_2^{2k+l,-k}(\mathcal{F}^\bullet, W_\bullet)$ are semistable vector bundles of slope $-k$ on $\mathbb{P}^1$. We get the same property for

$$E_1^{k,l}(\mathcal{F}^\bullet, W^B_\bullet) = H^{k+l}(Gr^{W^B}_{-k}\mathcal{F}^\bullet),$$

which is exactly the property required to say that $(\mathcal{F}^\bullet, W^B_\bullet)$ is a B-mixed twistor complex. The remaining statements of the lemma may be verified from the above discussion. □

Let MTC^D (resp. MTC^B) be the category of D-mixed (resp. B-mixed) twistor complexes. Notice that a complex in the category MTS in the category of mixed twistor structures, is in particular a B-mixed twistor complex. Thus we have a functor

$$\mathrm{Cpx}(MTS) \to MTC^B,$$

and Beilinson shows that this gives an equivalence of derived categories. There isn't a natural lift along the functor $Dec : MTC^D \to MTC^B$, but Dec also induces an equivalence of derived categories.

The difference between MTC^D and MTC^B may be seen in the loss of information going from MTC^D to MTC^B: a D-mixed twistor complex yields the associated $E_1^{2k+l,-k}$ terms of the spectral sequence, which are themselves semistable bundles of slope $-k$ on $\mathbb{P}^1$. However, the $E_0^{k,l}$-terms of the spectral sequence for the associated B-mixed twistor complex are extensions of these bundles by terms $\mathcal{U}^{k,l}$ of an acyclic complex. From here, one cannot in general recover the $E_1^{2k+l,-k}$ term of the original D-mixed twistor complex. It is a question of taste, how much one wants to consider this extra information as a part of the geometrical structure. For a given singular variety, if we choose different simplicial resolutions, the Deligne $E_1^{2k+l,-k}$-terms might be different. On the other hand, Deligne's E_2 terms, which are the same

as Beilinson's E_1 terms, are invariant as may be stated in the following corollary.

Corollary 10.4. *Suppose* $(\mathcal{F}^\bullet, W_\bullet) \xrightarrow{\phi} (\mathcal{G}^\bullet, W_\bullet)$ *is a morphism of D-mixed twistor complexes. Suppose that* ϕ *induces an isomorphism (resp. injection, resp. surjection) on cohomology* $\phi : H^i(\mathcal{F}^\bullet) \cong H^i(\mathcal{G}^\bullet)$. *Then the map induced by* ϕ

$$E_2^{k,l}(\mathcal{F}^\bullet, W_\bullet) \to E_2^{k,l}(\mathcal{G}^\bullet, W_\bullet)$$

is an isomorphism (resp. injection, resp. surjection) of pure vector bundles on $\mathbb{P}^1$.

Suppose $(\mathcal{F}^\bullet, W_\bullet^B) \xrightarrow{\phi} (\mathcal{G}^\bullet, W_\bullet^B)$ *is a morphism of B-mixed twistor complexes. Suppose that* ϕ *induces an isomorphism (resp. injection, resp. surjection) on cohomology* $\phi : H^i(\mathcal{F}^\bullet) \cong H^i(\mathcal{G}^\bullet)$. *Then the map induced by* ϕ

$$E_1^{k,l}(\mathcal{F}^\bullet, W_\bullet) \to E_1^{k,l}(\mathcal{G}^\bullet, W_\bullet)$$

is an isomorphism (resp. injection, resp. surjection) of pure vector bundles on $\mathbb{P}^1$.

Proof. In both cases, the indicated terms of the spectral sequence are equal to the associated graded pieces of the mixed twistor structure given by Corollaries 10.1 and 10.2. Strictness for maps between mixed twistor structures [99] says that injectivity and surjectivity pass to the associated-graded pieces. □

Suppose we are given a functor

$$G : \Delta \to MTC^D$$

denoted by $k \mapsto (G^\bullet(k), W_\bullet)$. Then Deligne defines the total complex

$$\mathbf{tot}(G)^j := \bigoplus_{i+k=j} G^i(k)$$

with weight filtration

$$W_m^{\mathrm{Dec}_1}\mathbf{tot}(G)^j := \bigoplus_{i+k=j} W_{m-k}G^i(k).$$

The differentials of $\mathbf{tot}(G)^\bullet$ are obtained by combining the differentials of $G^\bullet(k)$ with the alternating sums of the simplicial face maps.

If we are given a functor

$$G : \Delta \to MTC^B, \quad k \mapsto (G^\bullet(k), W_\bullet^B)$$

then Beilinson considers the same total complex with differential

$$\mathbf{tot}(G)^j := \bigoplus_{i+k=j} G^i(k)$$

but with weight filtration

$$W_m^B \mathbf{tot}^B(G)^j := \bigoplus_{i+k=j} W_m G^i(k).$$

Proposition 10.5. *For a cosimplicial D-mixed twistor complex G, the total complex $(\mathbf{tot}(G)^\bullet, W_\bullet^{\mathrm{Dec}_1})$ is again a D-mixed twistor complex, inducing a mixed twistor structure on $H^i(\mathrm{tot}(G)^\bullet)$.*

For a cosimplicial B-mixed twistor complex G, the total complex $(\mathbf{tot}(G)^\bullet, W_\bullet^B)$ is again a B-mixed twistor complex, inducing a mixed twistor structure on $H^i(\mathbf{tot}(G)^\bullet)$.

If we start with a cosimplicial D-mixed twistor complex G and let $W^B G(k)$ be the filtration of $G^\bullet(k)$ considered in Lemma 10.3, varying functorially in k to give a cosimplicial B-mixed twistor complex. Both of these induce the same mixed twistor structure on $H^i(\mathbf{tot}(G)^\bullet)$.

Proof. As in [33]. □

Remark 10.6. *Given a cosimplicial D-mixed twistor complex G, we get a D-mixed twistor complex $(\mathbf{tot}(G), W_\bullet^{\mathrm{Dec}_1})$ from the first paragraph of the proposition, then a B-mixed twistor complex by Lemma 10.3. On the other hand, applying the construction of Lemma 10.3 levelwise we get a cosimplicial B-mixed twistor complex, which also gives a B-mixed twistor complex by the second paragraph of the proposition. These will not in general be the same; the third paragraph of the proposition says that they still induce the same weight filtration on the total cohomology.*

Suppose X is a smooth projective variety. A polarizable pure variation of twistor structure of weight w (VTS) is just a semisimple local system L on X. The weight w may be chosen arbitrarily, and determines the realization of L into a family of twistor structures parametrized by $x \in X$, which we denote by L^w. See [99] [87].

Given a VTS L^w of weight w, we obtain a D-mixed twistor complex $(A^\bullet_{\mathrm{tw}}(X, L^w), W_\bullet)$ as follows. The weight filtration will be trivially concentrated in degree w, that is to say

$$W_m A^i_{\mathrm{tw}}(X, L^w) = \begin{cases} A^i_{\mathrm{tw}}(X, L^w) & m \geq w \\ 0 & m < w. \end{cases} \tag{10.2}$$

So we just have to define the complex $A^\bullet_{\mathrm{tw}}(X, L^w)$. Let $\mathcal{L}$ be the C^∞ bundle underlying the local system, and put

$$A^i_{\mathrm{tw}}(X, L^w) := A^i(X, L) \otimes_{\mathbb{C}} \mathcal{O}_{\mathbb{P}^1}(w + i).$$

Since L is semisimple, it has a structure of harmonic bundle [28] [37], giving a decomposition of the flat connection d on $\mathcal{L}$ into

$$d = \partial + \overline{\partial} + \theta + \overline{\theta}$$

in the notations of [95]. Let $\lambda, \mu : \mathcal{O}_{\mathbb{P}^1} \to \mathcal{O}_{\mathbb{P}^1}(1)$ denote the two sections vanishing respectivly at 0 and ∞. Then

$$d_{\mathrm{tw}} := \lambda(\partial + \overline{\theta}) + \mu(\overline{\partial} + \theta) = \lambda D' + \mu D''$$

defines an operator

$$d_{\mathrm{tw}} : A^i(X,L) \otimes_{\mathbb{C}} \mathcal{O}_{\mathbb{P}^1}(w+i) \to A^{i+1}(X,L) \otimes_{\mathbb{C}} \mathcal{O}_{\mathbb{P}^1}(w+i+1)$$

which is to say a differential for $A^{\bullet}_{\mathrm{tw}}(X, L^w)$.

The variation of twistor structure on L corresponds to a prefered section of the twistor moduli stack $\mathcal{M}_{DH}(X, GL(n))$, and the above complex is the Deligne-Hitchin glueing of the complexes calculating cohomology of λ-connections on X and $\overline{X}$, see [87] [99].

Lemma 10.7. *The complex $(A^{\bullet}_{\mathrm{tw}}(X, L^w), d_{\mathrm{tw}})$ together with the weight filtration $W_{\bullet}$ of* (10.2) *concentrated trivially in degree w, is a D-mixed twistor complex.*

Proof. See [99]. Recall from [95] that the cohomology of d is the same as that of $(\ker(\partial + \overline{\theta}), \overline{\partial} + \theta)$ or symmetrically $(\ker(\overline{\partial} + \theta), \partial + \overline{\theta})$, these cohomologies are isomorphic to the spaces of harmonic forms, and in fact there is a $D'D''$-lemma. From these, the cohomology bundle $H^i(A^{\bullet}_{\mathrm{tw}}(X, L^w), d_{\mathrm{tw}})$ is isomorphic to the cohomology of the sequence

$$A^{i-2}(X,L) \stackrel{D'D''}{\to} A^i(X,L) \stackrel{(D',D'')}{\to} A^{i+1}(X,L) \oplus A^{i+1}(X,L),$$

all tensored with $\mathcal{O}_{\mathbb{P}^1}(w+i)$. Hence

$$H^i(A^{\bullet}_{\mathrm{tw}}(X, L^w), d_{\mathrm{tw}}) = H^i(X, L) \otimes \mathcal{O}_{\mathbb{P}^1}(w+i).$$

The D-mixed twistor property follows immediately. $\square$

We now complete the twistor analogue of the main construction of Hodge III [33]. If $L_{\bullet}$ is a local system on a simplicial smooth projective variety $X_{\bullet}$ such that each L_k is a semisimple local system on X_k, then for any integer w $L_{\bullet}$ has a structure of polarizable variation of pure twistor structure of weight w denoted $L^w_{\bullet}$.

Corollary 10.8. *In this situation, the cohomology $H^i(X_{\bullet}, L^w_{\bullet})$ has a natural mixed twistor structure whose underlying bundle over $\mathbb{P}^1$ is obtained by the Deligne-Hitchin glueing.*

This mixed twistor structure is functorial for morphisms between local systems, compatible with cup-product, and contravariantly functorial for morphisms of simplicial varieties in the following way. Suppose $f : X_{\bullet} \to Y_{\bullet}$ is a morphism of simplicial smooth projective varieties,

and that $L_\bullet$ is a local system on $Y_\bullet$ with each L_k semisimple. Fix a weight w. Then f induces a map of mixed twistor structures

$$H^i(Y_\bullet, L_\bullet^w) \to H^i(X_\bullet, f^*(L)_\bullet^w).$$

If the map on cohomology is an isomorphism then it is an isomorphism of mixed twistor structures.

Proof. The D-mixed twistor complexes of Lemma 10.7 are contravariantly functorial, so they fit together into a cosimplicial D-mixed twistor complex. (Complexes of forms on simplicial manifolds are discussed in [39] [55].) By Proposition 10.5 this gives a total D-mixed twistor complex inducing a mixed twistor structure on cohomology. Functoriality follows from the construction and the last phrase comes from the strictness property for mixed twistor structures [99]. □

Corollary 10.9. *Suppose X is a connected smooth proper DM-stack. If L is semisimple local system considered as a pure variation of twistor structure of weight w, then the mixed twistor structure on $H^i(X, L)$ is pure of weight $i + w$.*

Proof. Choose a dominant morphism from a smooth projective variety $p : Z \to X$. By Corollary 9.11, the morphism on cohomology

$$H^i(X, L) \to H^i(Z, p^*(L))$$

is injective. This is a morphism of mixed twistor structures and the one on the right is pure, so the one on the left is pure too. □

It would clearly be interesting to develop a theory of variations of mixed twistor structures over simplicial varieties, leading to a mixed twistor structure on the total cohomology. This would go beyond our present scope; but see [99] for a discussion of VMTS on a single smooth variety.

11. Finite group actions

In this section, we discuss some examples which may be obtained by considering finite group actions.

Suppose Φ is a finite group acting on a connected smooth projective variety X, and G is a complex linear algebraic group. Then Φ acts on the moduli stacks $\mathcal{M}_\eta(X, G)$ preserving all of the various structures.

Lemma 11.1. *In the above situation, the substack of stacky fixed points is identified with the moduli stack for the quotient $Y = X/\!/\Phi$:*

$$\mathcal{M}_\eta(X, G)^\Phi \cong \mathcal{M}_\eta(Y, G).$$

Proof. A stacky fixed point in $\mathcal{M}_\eta(X,G)$ is defined as an object together with a compatible action of Φ covering the action on X, which is exactly the same as an object with descent data down to Y. □

One can observe that for a global quotient stack, the fundamental group is also the fundamental group of a smooth projective variety, and the fixed point stack of the preceding lemma can be interpreted in this way. This observation was already present in Daskalopoulos-Wentworth [29].

Proposition 11.2. *If $Y = X/\!/\Phi$ is a global quotient stack for a group Φ acting on a connected smooth projective variety X, then we can construct a connected smooth projective variety Z and a map $f : Z \to Y$ inducing an isomorphism $\pi_1(Z,z) \cong \pi_1(Y, f(z))$. In particular, $\mathcal{M}_\eta(Y,G) \cong \mathcal{M}_\eta(Z,G)$.*

Proof. Indeed, there exists a smooth projective variety U with $\pi_1(U) = G$ by Serre's construction [91], see also Browder and Katz [22]. Let P be the universal cover of U, so P is simply connected and has a free action of G. Now put

$$Z := Y \times P/G.$$

This is a smooth projective variety provided with a map $f : Z \to Y$ which is a fiber bundle in the etale topology of Y. The fiber P is simply connected fiber, so the long exact sequence of homotopy groups implies that f induces an isomorphism on π_1. □

A more subtle question concerns the quotient of the group action. The group action preserves all of the structure on the moduli stack, hence for example the subset of smooth points of the moduli space quotient $M_\eta(X,G)/\Phi$ admits a hyperkähler structure. This suggests that $\mathcal{M}_\eta(X,G)/\!/\Phi$ should itself be viewed as a kind of "nonabelian 1-motive". We look at how to realize it as a connected component of a moduli stack.

Consider first the case where a finite group Φ acts on a group G but acts trivially on X. Let $H = G \rtimes \Phi$ be the semidirect product fitting into the split exact sequence

$$1 \to G \to H \to \Phi \to 0.$$

This induces a sequence of maps of moduli stacks

$$\mathcal{M}_\eta(X,G) \to \mathcal{M}_\eta(X,H) \to \mathcal{M}_\eta(X,\Phi).$$

The trivial Φ-torsor has Φ as group of automorphisms, so it corresponds to a map

$$B\Phi \to \mathcal{M}_\eta(X,\Phi).$$

Lemma 11.3. *With the above notations we have a cartesian square of moduli stacks for $\eta = B, DR, H \ldots$ refering to any type of local system*

$$\begin{array}{ccc} \mathcal{M}_\eta(X,G)/\!/\Phi & \to & \mathcal{M}_\eta(X,H) \\ \downarrow & & \downarrow \\ B\Phi & \to & \mathcal{M}_\eta(X,\Phi). \end{array}$$

Suppose now that Φ acts on our smooth projective variety X with DM-stack quotient $Y := X/\!/\Phi$; in fact X could also be a DM-stack itself.

Let $G \wr \Phi$ denote the wreath product (these have been used for geometry, cf [113]), that is the semidirect product of Φ with its permutation action on $\prod_\Phi G$. Elements are denoted $(v, (g_w)_{w\in\Phi})$. There is a canonically split projection $G \wr \Phi \to \Phi$, which induces a map on moduli spaces.

There is an action of Φ on $G \wr \Phi$, combining its adjoint action on itself, its translation action on $\prod_{w\in\Phi} G$, and its given action on G. The formula is

$$\varphi \in \Phi : (v, (g_w)_{w\in\Phi}) \mapsto (\varphi v \varphi^{-1}, (\varphi(g_{\varphi^{-1}w}))_{w\in\Phi}).$$

Let $H := (G \wr \Phi) \rtimes \Phi$ be the semidirect product for this action.

The covering $X \to Y$ is a Φ-torsor which induces a point denoted

$$[X] = * \to \mathcal{M}_\eta(Y,\Phi)$$

in any of the moduli spaces of Φ-local systems over Y, which all parametrize Φ-torsors since Φ is a finite group.

Use first this torsor and the group $G \wr \Phi$ to transform the action of Φ on $\mathcal{M}_\eta(X,G)$ to an action on the group only, the case of Lemma 11.3.

Proposition 11.4. *Let $Y := X/\!/\Phi$ be the DM-stack quotient. Then for any type of local system η we have a cartesian diagram of moduli stacks*

$$\begin{array}{ccc} \mathcal{M}_\eta(X,G) & \to & \mathcal{M}_\eta(Y, G \wr \Phi) \\ \downarrow & & \downarrow \\ [X] & \to & \mathcal{M}_\eta(Y,\Phi). \end{array}$$

This is compatible with the action of Φ, given on X and G, thereby induced on $G \wr \Phi$, and by the adjoint action on Φ for the lower right corner.

Proof. If P is a principal G-bundle over X, a group element $w \in \Phi$ tranlates it to a new one w^*P defined by $(w^*P)_x := P_{w^{-1}x}$. We get a principal $\prod_{w\in\Phi} G$-bundle over X

$$\prod_{w\in\Phi} w^*P \to X,$$

but Φ also acts on this bundle so it may be considered as a principal $G \wr \Phi$-bundle over Y. This construction respects structures of flat λ-connection or a structure of topological local system, so it defines a map

$$\mathcal{M}_\eta(X, G) \to \mathcal{M}_\eta(Y, G \wr \Phi).$$

The image under the map $G \wr \Phi \to \Phi$ which in our notations is just projection to the first coordinate, is naturally isomorphic to the covering X considered as a Φ-torsor. This completes the construction of the commutative square in the proposition. It is compatible with the various actions of Φ.

To finish the proof we have to show that it is cartesian. Suppose Q is a $G \wr \Phi$-bundle over Y, projecting to a Φ-torsor provided with an isomorphism to X. This gives a map $Q \to X$ which is a $\prod_{w\in\Phi} G$-torsor over X. Changing structure group by the projection at the identity element

$$\prod_{w\in\Phi} G \to G, \quad (g_w)_{w\in\Phi} \mapsto g_1$$

yields a G-torsor P over X. This construction provides the required isomorphism between $\mathcal{M}_\eta(X, G \wr \Phi)$ and the fiber product in the cartesian square. $\square$

The semidirect product $H = (G \wr \Phi) \rtimes \Phi$ fits into an exact sequence

$$1 \to \prod_{w\in\Phi} G \to H \to (\Phi \rtimes \Phi) \to 1$$

where the quotient is the semidirect product made using the adjoint action of Φ on itself. The Φ-torsor X yields by extension of structure group a $\Phi \rtimes \Phi$-torsor $X \times^{\Phi} (\Phi \rtimes \Phi)$ which projects to the trivial Φ-torsor under the quotient map $\Phi \rtimes \Phi \to \Phi$. The group Φ acts by automorphisms on $X \times^{\Phi} (\Phi \rtimes \Phi)$, giving a map to the moduli stack

$$B\Phi \to \mathcal{M}_\eta(X, \Phi \rtimes \Phi). \tag{11.1}$$

Corollary 11.5. *If Φ acts on G and X, setting $Y := X/\!/\Phi$ and $H := (G \wr \Phi) \rtimes \Phi$, we have a cartesian square of algebraic stacks*

$$\begin{array}{ccc} \mathcal{M}_\eta(X, G)/\!/\Phi & \to & \mathcal{M}_\eta(Y, H) \\ \downarrow & & \downarrow \\ B\Phi & \to & \mathcal{M}_\eta(X, \Phi \rtimes \Phi). \end{array}$$

Proof. Proposition 11.4 allows us to express the action of Φ on the moduli stack $\mathcal{M}_\eta(X, G)$ as coming from an action on the group $G \wr \Phi$ only, for local systems over the DM-stack quotient $Y = X/\!/\Phi$. Lemma 11.3 then gives the stack quotient of the moduli space as a pullback

over $B\Phi$. Combining the two pullbacks amounts to taking the pullback over the map (11.1). $\square$

This corollary motivates the introduction of DM-stacks for looking at group actions on the moduli of local systems over a smooth projective variety X.

Question 11.6. *What are the properties of the induced square*

$$\begin{array}{ccc} M_\eta(X,G)/\Phi & \to & M_\eta(Y,H) \\ \downarrow & & \downarrow \\ * & \to & M_\eta(Y, \Phi \rtimes \Phi). \end{array}$$

of coarse moduli spaces?

The moduli space $M_\eta(X, \Phi \rtimes \Phi)$ is discrete. It would be good to be able to say that $M_\eta(X,G)/\Phi$ is identified as an irreducible component of $M_\eta(Y,H)$ but that seems to be a perhaps somewhat delicate question about character varieties.

12. Fundamental groups of irreducible varieties

Many years ago, Domingo Toledo asked the following question: is every finitely presented group the fundamental group of an irreducible singular variety? In this section we give a streamlined argument to show that the answer is 'yes'.

Take note of the following construction. Suppose X is quasiprojective, Z a closed subscheme, and $r : Z \to Y$ a finite morphism. Then there is a scheme W obtained by "contracting along r". More precisely, W is provided with a morphism $p : X \to W$ and a factorization

$$\begin{array}{ccc} Z & \hookrightarrow & X \\ \downarrow & & \downarrow \\ Y & \to & W \end{array}$$

which is universal, that is to say it is a cocartesian square in the category of schemes. Furthermore $Y \hookrightarrow W$ is a closed embedding, the above square is also cartesian, W is separated of finite type over $\mathbb{C}$, and the morphism p is finite. The coproduct may be denoted by

$$W = X/r = X \cup^Z Y.$$

The associated diagram of topological spaces

$$\begin{array}{ccc} Z^{\mathrm{top}} & \hookrightarrow & X^{\mathrm{top}} \\ \downarrow & & \downarrow \\ Y^{\mathrm{top}} & \to & W^{\mathrm{top}} \end{array}$$

is also cocartesian and cartesian.

From this we get the Brown-Van Kampen statement: that for any $0 \leq n \leq \infty$ the diagram of n-groupoids

$$\begin{array}{ccc} \Pi_n(Z^{\rm top}) & \rightarrow & \Pi_n(X^{\rm top}) \\ \downarrow & & \downarrow \\ \Pi_n(Y^{\rm top}) & \rightarrow & \Pi_n(W^{\rm top}) \end{array}$$

is cocartesian in the $n+1$-category of n-groupoids. For $n = \infty$ this just says that the previous diagram of spaces is a homotopy pushout.

For $n = 1$, the diagram of fundamental groupoids

$$\begin{array}{ccc} \Pi_1(Z^{\rm top}) & \rightarrow & \Pi_1(X^{\rm top}) \\ \downarrow & & \downarrow \\ \Pi_1(Y^{\rm top}) & \rightarrow & \Pi_1(W^{\rm top}) \end{array}$$

is a cocartesian diagram in the 2-category of groupoids.

Theorem 12.1. *Suppose Υ is a finitely presented group. Then there is an irreducible projective variety W with $\pi_1(W^{\rm top}) \cong \Upsilon$.*

Proof. Suppose Υ is a finitely presented group. It may be realized as the fundamental group of a 2-dimensional simplicial complex A. Here A consists of a set of vertices, plus a subset of pairs of vertices called the edges, and a subset of triples of vertices called the triangles, such that the edges of the triangles are contained in the set of edges. Such a complex A is realized into a topological space $|A|$ in an obvious way.

We furthermore may assume that every vertex is contained in some edge, every edge is contained in some triangle, and the set of triangles is connected by the adjacency relation (two triangles being adjacent if they share the same edge).

Let G be the dual graph whose points are the triangles, and whose edges are the edges common to two triangles. Choose a maximal tree $T \subset G$. This determines a set of edges of A. Define the unfolding of A along T denoted by $\tilde{A}$ to be the simplicial complex formed by the triangles of A joined together along only those edges corresponding to elements of T.

Observe that the topological realization $|\tilde{A}|$ is simply connected, being a union of triangles inductively joined along single edges according to the tree pattern. On the other hand, the 1-skeleta are 1-dimensional simplicial complexes provided with a map preserving the structure of simplicial complex

$$\tilde{A}_1 \rightarrow A_1$$

which induces a map on realizations

$$|\tilde{A}_1| \rightarrow |A_1|.$$

The diagram of spaces

$$\begin{array}{ccc} |\tilde{A}_1| & \to & |\tilde{A}| \\ \downarrow & & \downarrow \\ |A_1| & \to & |A| \end{array}$$

is cocartesian, so the corresponding diagram of fundamental groupoids

$$\begin{array}{ccc} \Pi_1(|\tilde{A}_1|) & \to & \Pi_1(|\tilde{A}|) \\ \downarrow & & \downarrow \\ \Pi_1(|A_1|) & \to & \Pi_1(|A|) \end{array}$$

is cocartesian. Note however that $\Pi_1(|\tilde{A}|) = *$ is trivial and $\Pi_1(|A|)$ is equivalent to the group Υ. Thus Υ is expressed as the homotopy contraction of $\Pi_1(|A_1|)$ along $\Pi_1(|\tilde{A}_1|)$.

Now A_1 and $\tilde{A}_1$ are just graphs and the map preserves the edge structure. Hence there are configurations of lines Y and Z, that is to say $Y = \bigcup Y_i$ and $Z = \bigcup Z_j$ with $Y_i \cong \mathbb{P}^1$ and $Z_j \cong \mathbb{P}^1$, such that the Y_i correspond to edges of A_1 meeting at points corresponding to the vertices of A_1, and the Z_j correspond to edges of $\tilde{A}_1$ meeting at points corresponding to the vertices of $\tilde{A}_1$. The map $\tilde{A}_1 \to A_1$ corresponds to a finite map $Z \to Y$. We obtain a commutative diagram

$$\begin{array}{ccc} \Pi_1(|\tilde{A}_1|) & \to & \Pi_1(Z^{\mathrm{top}}) \\ \downarrow & & \downarrow \\ \Pi_1(|A_1|) & \to & \Pi_1(Y^{\mathrm{top}}) \end{array}$$

where the horizontal arrows are equivalences of groupoids.

Embedd now Z in a projective space X, and let W be the quotient obtained by contracting X along $Z \to Y$. As $\Pi_1(X^{\mathrm{top}}) \sim *$, it follows that the diagram

$$\begin{array}{ccc} \Pi_1(Z^{\mathrm{top}}) & \to & \Pi_1(X^{\mathrm{top}}) \\ \downarrow & & \downarrow \\ \Pi_1(Y^{\mathrm{top}}) & \to & \Pi_1(W^{\mathrm{top}}) \end{array}$$

is the same as

$$\begin{array}{ccc} \Pi_1(|\tilde{A}_1|) & \to & \Pi_1(|\tilde{A}|) \sim * \\ \downarrow & & \downarrow \\ \Pi_1(|A_1|) & \to & \Pi_1(|A|) \sim \Upsilon. \end{array}$$

Thus $\pi_1(W^{\mathrm{top}}) \cong \Upsilon$, and W is irreducible by construction. □

Question 12.2. *Is it possible to construct an irreducible variety W with $\pi_1(W) \cong \Upsilon$, such that the singularities of W are normal crossings?*

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CNRS, Laboratoire J. A. Dieudonné, UMR 6621, Université de Nice-Sophia Antipolis, 06108 Nice, Cedex 2, France
E-mail address: carlos@unice.fr
URL: http://math.unice.fr/∼carlos/